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CERTAIN FRACTIONAL OPERATORS OF EXTENDED MITTAG-LEFFLER FUNCTION

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Abstract. The aim of this paper is to deal with the extension of Mittag-Leffler function. Composition of fractional calculus operators of integration and differentiation of complex order known as Marichev-Saigo-Maeda operators containing Appell's function F_3 in its kernel with the proposed function is defined. Further, corollaries and consequences are the special cases of our main results.

1. Introduction

Fractional calculus, subject of integrals and differentials of arbitrary orders, has a number of applications in the field of applied sciences, engineering, physics, statistics and mathematics, involving various special functions. During the past two decades, researchers are interested in extension and generations of special functions due to ability of diversity.

Srivastava et. al. introduced a function [29]

$$\Theta\left(\{\kappa_n\}_{n\in N_0};z\right):=\left\{\begin{array}{cc}\sum_{n=0}^{\infty}\kappa_n\frac{z^n}{n!} & (|z|<\Re;0<\Re<\infty;\kappa_0:=1)\\ m_0z^{\varpi}\exp(z)[1+O\left(\frac{1}{z}\right)] & (\Re(z)\to\infty;m_0>0;\varpi\in C)\end{array}\right\} \tag{1.0.1}$$

where m_0 and ϖ are some constants depending upon $\{\kappa_n\}_{n\in N_0}$ where $\{\kappa_n\}_{n\in N_o}$ is a bound sequence. Thus, by the function $\Theta\left(\{\kappa_n\}_{n\in N_0}\right)$, corresponding generalized extended form of Gamma function, Beta function and Gauss hypergeometric function are as follows.

$$\Gamma_p^{\left(\{\kappa_n\}_{n\in N_0}\right)}(z):=\int_0^{\infty}s^{z-1}\Theta\left(\{\kappa_n\}_{n\in n_0};-s-\frac{p}{s}\right)ds \tag{1.0.2}$$

$$(\Re(z)>0,(\Re(p)\geq 0)$$

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$$B_p^{(\{\kappa_n\}_{n\in N_0})}(\alpha,\beta;p) := \int_0^\infty s^{\alpha-1}(1-s)^{\beta-1}\Theta\left(\{\kappa_n\}_{n\in N_0}; -\frac{p}{s(1-s)}\right) ds, \quad (1.0.3)$$

$$(mim\{\Re(\alpha), \Re(\beta)\} > 0, \Re(p) \geq 0)$$

$$\Im_p^{(\{\kappa_n\}_{n\in N_0})}(a,b;c;z) := \sum_{k=0}^{\infty} (a)_k \frac{B_p^{(\{\kappa_n\}_{n\in N_0})}(b+k, c-b;p)}{B(b,c-b)} \frac{z^k}{k!} \quad (1.0.4)$$

$$(|z| < 1; \Re(c) > \Re(b) > 0; \Re(p) \geq 0)$$

Regarding function $\Theta(\{\kappa_n\}_{n\in N_0})$, Parmar [17] proposed the extension of Mittag-Leffler function by means of extended Beta function (1.0.3).

$$E_{\xi,\zeta}^{(\{\kappa_n\};\gamma)}(z;p) := \sum_{k=0}^{\infty} \frac{B_p^{(\{\kappa_n\}_{n\in N_0})}(\gamma+k, 1-\gamma;p)}{B(\gamma,1-\gamma)} \frac{z^k}{\Gamma(\xi k+\zeta)} \quad (1.0.5)$$

$$(z, \zeta, \gamma \in C; \Re(\xi) > 0, \Re(\zeta > 0, \Re(\gamma) > 1; p \geq 0)$$

This is basically extension of exponential function.

2. Special Cases:

As the generalization of functions depends upon the bounded sequence, here we discuss some particular cases.

1) If we chose $\kappa_n = 0$ then the definition of extended Mittag-Leffler function immediately reduces to Probhakar function, which is a function of three parameters [19].

$$E_{\xi,\zeta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\xi k+\zeta)} \frac{z^k}{k!} \quad (2.0.6)$$

$$(\xi, \zeta, \gamma \in C; \Re(\xi) > 0, \Re(\zeta) > 0)$$

2) On the selection of $\kappa_n = 1$ for the sequence then (1.0.2),(1.0.3) and (1.0.4) reduces to the forms defined by Chaudhry et al. [1], Chaudhry et al. [2] and Chaudhry and Zubair [3] and (1.0.5) takes the extended form introduced and systematically consider by Özarslan and Yilmaz [16] and it takes the form.

$$E_{\xi,\zeta}^{\gamma}(z;p) = \sum_{k=0}^{\infty} \frac{B(\gamma+k, 1-\gamma;p)}{B(\gamma,1-\gamma)} \frac{z^k}{\Gamma(\xi k+\zeta)} \quad (2.0.7)$$

$$(\xi, \zeta, \gamma \in C; \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\delta) > 1; p \geq 0)$$

3) If we select $\kappa_n = \frac{(\mu)_n}{(\nu_n)}$ then the extended Gamma function (1.0.2), extended Beta function (1.0.3) and extended Gauss hypergeometric function (1.0.4) reduce to the form introduced by Özarslan et al. [16] and proposed extended Mittag-Leffler function takes the form

$$E_{\xi,\zeta}^{(\mu,\nu);\gamma}(z;p) := \sum_{k=0}^{\infty} \frac{B^{(\mu,\nu)}(\gamma+k, 1-\gamma;p)}{B(\gamma, 1-\gamma)} \frac{z^k}{\Gamma(\xi k+\zeta)} \tag{2.0.8}$$

$$(z, \zeta, \gamma \in C; \Re(\mu) > 0, \Re(\nu) > 0, \Re(\xi) > 0, \Re(\gamma) > 1; p \geq 0)$$

4) Further, on the selection of $\xi = \zeta = 1$ the proposed extended Mittag-Leffler function and functions defined in (1.0.5),(2.0.8) and (2.0.6) becomes extended confluent hypergeometric function.

$$E_{1,1}^{(\{\kappa_n\};\gamma)}(z;p) = \Phi_p^{(\{\kappa_n\}_{n\in N_0})}(\gamma;1;z) \tag{2.0.9}$$

$$E_{1,1}^{(\mu,\nu);\gamma}(z;p) = \Phi_p^{(\mu,\nu)}(\gamma;1;z) \tag{2.0.10}$$

and

$$E_{1,1}^{\gamma}(z;p) = \Phi_p(\gamma;1;z) \tag{2.0.11}$$

In this sequel, we get composite function under the influence of integral and differential of arbitrary order. Thus, in order to decompose the function, we need the concept of Hadamard product which gives help to decompose a function into two known analytical functions.

Definition 2.0.1. [18] Let $g(z) := \sum_{k=0}^{\infty} \alpha_k z^k$ and $h(z) := \sum_{k=0}^{\infty} \beta_k z^k$ be two power series having R_g and R_h radius of convergence of the power series then Hadamard product is defined as

$$(g * h)(z) := \sum_{k=0}^{\infty} \alpha_k \beta_z z^k \tag{2.0.12}$$

The radius of convergence R of the Hadmard product series satisfies $R_g.R_h \leq R$. Hadamard product fulfills the condition that if one series is an entire function then Hadamard product series $(g*h)(z)$ is also an entire function. see [18], [24], [32] for further detail and the references cited therein.

The aim of this paper is to study the composition of extended proposed function with fractional operators and to decompose newly-emerged function in terms of Hadamard product of extended Mittag-Leffler function and the Fox-Wright function.

Definition 2.0.2. Fox-Wright generalized hypergeometric function is defined as [3]

$${}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{array}{c}(\alpha_1, A_1)...(\alpha_p, A_p)\\(\beta_1, B_1)...(\beta_q, B_q)\end{array}; z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i + kA_i)}{\prod_{j=1}^{q}\Gamma(\beta_j + kB_j)} \frac{z^k}{k!} \tag{2.0.13}$$

$$(\alpha_i \in C, A_i \in \Re\backslash 0 (j = 1, ..., p); \beta_j \in C, B_j \in \Re\backslash 0 (j = 1, ..., q); \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > -1)$$

where $(\alpha_1, ..., \alpha_p)$ are p numerator and $(\beta_1, ..., \beta_q)$ are q denominator parameters.

3. Fractional calculus of the extended Mittag-Leffler function

In this section, we establised some fractional integral and derivative formulas for the class of extended Mittag-Leffler function, given in (1.0.5). We now recall our main fractional derivative operators $D_{0+}^{\omega,\epsilon,\eta}$ and $D_{0-}^{\omega,\epsilon,\eta}$. These differential operators are defined in terms of the corresponding pairs of hypergeometric fractional integral operators $I_{0+}^{\omega,\epsilon,\eta}$ and $I_{0-}^{\omega,\epsilon,\eta}$ respectively known as Saigo operators involving Gauss hypergeometric function ${}_2F_1$ as the kernel,(see,for details [21]).Srivasatava and Saigo [30] systematically presented their compositional and ther properties as well as their applications to some boundary value problems involving the Euler-Darboux equation.

Definition 3.0.3. The left-sided fractional integral operator and corresponding left-sided fractional differential operator involving Gauss hypergeometric function ${}_2F_1$ as its kernel are defined, for $x > 0$ and $\omega, \epsilon, \eta \in C$, by

$$\left(I_{0+}^{\omega,\epsilon,\eta} f\right)(x) = \frac{x^{-\omega-\epsilon}}{\Gamma(\omega)} \int_0^x (x-t)^{\omega-1} {}_2F_1\left(\omega+\epsilon, -\eta; \omega; 1-\frac{t}{x}\right) f(t)dt \qquad (3.0.14)$$

$$(\Re(\omega) > 0)\,;$$

$$= \frac{d^k}{dx^k}\left(I_{0+}^{\omega+k,\epsilon-k,\eta-k} f\right)(x) \quad (\Re(\omega) \leq 0; k = [\Re(-\omega)] + 1)\,; \qquad (3.0.15)$$

and

$$\left(D_{0+}^{\omega,\epsilon,\eta} f\right)(x) = \left(I_{0+}^{-\omega,-\epsilon,\omega+\eta} f\right)(x) = \frac{d^k}{dx^k}\left(I_{0+}^{-\omega+k,-\epsilon-k,\omega+\eta-k} f\right)(x) \qquad (3.0.16)$$

$$(\Re(\omega) > 0; k = [\Re(-\omega)] + 1);$$

where ${}_2F_1(.)$ is Gaussian hypergeometric function and is defined as [20].

$${}_2F_1(\alpha, \beta; \chi; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\chi)_k} \frac{x^k}{k!} \qquad (3.0.17)$$

where $\epsilon = -\omega$, left-sided hypergeometric fractional integral operator $I_{0+}^{\omega,\epsilon,\eta}$ and hypergeometric fractional differential operator $D_{0+}^{\omega,\epsilon,\eta}$ coincide with classical Riemann-Liouville integral $R_{0,x}^{\omega}$ and Riemann-Liouville differential operator ${}_{RL}D_{0+}^{\omega}$ of order $\omega \in C$. Unify relationship shown below

$$(R_{0,x}^{\omega} f)(x) = (I_{0+}^{\omega,-\omega,\eta} f)(x) \qquad (3.0.18)$$

and

$$({}_{RL}D_{0+}^{\omega} f)(x) = (D_{0+}^{\omega,-\omega,\eta} f)(x) \qquad (3.0.19)$$

where

$$(R_{0,x}^{\omega} f)(x) = (I_{0+}^{\omega,-\omega,\eta} f)(x) = \frac{1}{\Gamma(\omega)} \int_0^x (x-t)^{\omega-1} f(t)dt, \;\; (\Re(\omega) > 0) \qquad (3.0.20)$$

$$= \frac{d^k}{dx^k}(R_{0,x}^{\omega+k} f)(x) \qquad (3.0.21)$$

$$(0 < \Re(\omega) + k \leq 1; k = 1, 2, ...);$$

$$\left({}_{RL}D_{0+}^{\omega}f\right)(x)=\left(D_{0+}^{\omega,-\omega,\eta}f\right)(x)=\left(\frac{d}{dx}\right)^{k}\frac{1}{\Gamma(k-\omega)}\int_{0}^{x}\frac{f(t)dt}{(x-t)^{\omega-k+1}} \quad (3.0.22)$$

$$(k=[\Re(\omega)]+1);$$

While for $\epsilon=0$ fractional integral operator $I_{0+}^{\omega,\epsilon,\eta}$ and fractional differential operator $D_{0+}^{\omega,\epsilon,\eta}$ in terms of Gauss hypergeometric function coincide with Erdelyi-Kober fractional integral $EK_{0,x}^{\omega,\eta}$ and differential operator ${}_{EK}D_{0+}^{\omega,\eta}$ of order $\omega\in C$ and unify relationship shown below.

$$(EK_{0,x}^{\omega,\eta}f)(x)=(I_{0+}^{\omega,0,\eta}f)(x) \quad (3.0.23)$$

and

$$({}_{EK}D_{0+}^{\omega,\eta}f)(x)=(D_{0+}^{\omega,0,\eta}f)(x) \quad (3.0.24)$$

where

$$\left(EK_{0,x}^{\omega,\eta}f\right)(x)=(I_{0+}^{\omega,0,\eta}f)(x)=\frac{x^{-\omega-\eta}}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\omega-1}t^{\eta}f(t)dt \quad (3.0.25)$$

$$(\Re(\omega)>0);$$

and

$$\left({}_{EK}D_{0+}^{\omega,\eta}f\right)(x)=x^{\eta}\left(\frac{d}{dx}\right)^{k}\left\{\frac{1}{\Gamma(k-\omega)}\int_{0}^{x}\frac{t^{\omega+\eta}f(t)}{(x-t)^{\omega-k+1}}dt\right\} \quad (3.0.26)$$

$$(x>0;k=[\Re(\omega)]+1;\Re(\omega)\geq 0);$$

The right-sided fractional integral operator $I_{0-}^{\omega,\epsilon,\eta}$ and corresponding right-sided fractional differential operator $D_{0-}^{\omega,\epsilon,\eta}$ involving Gauss hypergeometric function ${}_2F_1$ as the kernel are defined, for $x>0$ and $\omega,\epsilon,\eta\in C$, by

$$\left(I_{0-}^{\omega,\epsilon,\eta}f\right)(x)=\frac{1}{\Gamma(\omega)}\int_{x}^{\infty}(t-x)^{\omega-1}t^{-\omega-\epsilon}{}_2F_1\left(\omega+\epsilon,-\eta;\omega;1-\frac{x}{t}\right)f(t)dt \quad (3.0.27)$$

$$(\Re(\omega)>0);$$

$$=(-1)^{k}\frac{d^{k}}{dx^{k}}\left(I_{0-}^{\epsilon+k,\omega-k,\eta}f\right)(x) \quad (3.0.28)$$

$$(\Re(\omega)\leq 0;k=[\Re(-\omega)]+1);$$

and

$$\left(D_{0-}^{\omega,\epsilon,\eta}f\right)(x)=\left(I_{0-}^{-\omega,-\epsilon,\omega+\eta}\right)=(-1)^{k}\frac{d^{k}}{dx^{k}}\left(I_{0-}^{-\omega+k,-\epsilon-k,\omega+\eta}f\right)(x) \quad (3.0.29)$$

$$(\Re(\omega)>0;k=[\Re(-\omega)]+1);$$

When $\epsilon=-\omega$ the right-sided hypergeometric fractional integral operator $I_{0-}^{\omega,\epsilon,\eta}$ and the right-sided hypergeometirc differential operator $D_{0-}^{\omega,\epsilon,\eta}$ coincide with Weyl fractional integral $W_{x,\infty}^{\omega}$ and right-sided Weyl fractional derivative operator ${}_{W}D_{x,\infty}^{\omega}$ of order $\omega>0$ and unify relationship is shown below.

$$\left(W_{x,\infty}^{\omega}f\right)(x)=\left(I_{0-}^{\omega,-\omega,\eta}f\right)(x) \quad (3.0.30)$$

and

$$\left({}_{W}D_{\infty-}^{\omega}f\right)(x)=(D_{\infty-}^{\omega,-\omega,\eta}f)(x) \quad (3.0.31)$$

$$\left(W_{x,\infty}^{\omega}f\right)(x)=\left(I_{0-}^{\omega,-\omega,\eta}f\right)(x) \quad (3.0.32)$$

$$= \frac{1}{\Gamma(\omega)} \int_x^{\infty} (t-x)^{\omega-1} f(t)dt, \quad (\Re(\omega) > 0);$$

$$= (-1)^k \frac{d^k}{dx^k} (W_{x,\infty}^{\omega+k} f)(x), \quad (0 < \Re(\omega) + k \leq 1; k = 1, 2, ...) \tag{3.0.33}$$

and

$$({}_W D_{0-}^{\omega} f)(x) = \left(D_{0-}^{\omega,-\omega,\eta} f\right)(x) = \left(\frac{d}{dx}\right)^k \frac{(-1)^k}{\Gamma(k-\omega)} \int_x^{\infty} \frac{f(t)dt}{(t-x)^{\omega-k+1}} \tag{3.0.34}$$

$$(k = [\Re(\omega)] + 1);$$

And for $\epsilon = 0$ the right-sided hypergeometric fractional integral operator $I_{0-}^{\omega,\epsilon,\eta}$ and the right-sided hypergeometric fractional differential operator $D_{0-}^{\omega,\epsilon,\eta}$ coincide with Erdelyi-Kober fractional integral $EK_{x,\infty}^{\omega,\eta}$ and right-sided Erdelyi-Kober fractional derivative operator ${}_{EK}D_{x,\infty}^{\omega,\eta}$ of order $\omega > 0$ and unify relationship is shown below.

$$\left(EK_{x,\infty}^{\omega,\eta} f\right)(x) = \left(I_{\infty-}^{\omega,0,\eta} f\right)(x) \tag{3.0.35}$$

and

$$\left({}_{EK}D_{\infty-}^{\omega,\eta} f\right)(x) = \left(D_{\infty-}^{\omega,0,\eta} f\right)(x) \tag{3.0.36}$$

where (see, for details, Samko et al. [23])

$$\left(EK_{x,\infty}^{\omega,\eta} f\right)(x) = \left(I_{0-}^{\omega,0,\eta} f\right)(x) \tag{3.0.37}$$

$$= \frac{x^{\eta}}{\Gamma(\omega)} \int_x^{\infty} (t-x)^{\omega-1} t^{-\omega-\eta} f(t)dt; \quad (\Re(\omega) > 0);$$

and

$$\left({}_{EK}D_{\infty-}^{\omega} f\right)(x) = x^{\omega+\eta} \left(\frac{d}{dx}\right)^k \left\{\frac{1}{\Gamma(k-\omega)} \int_x^{\infty} \frac{t^{-\eta} f(t)}{(t-x)^{\omega-k+1}} dt\right\} \tag{3.0.38}$$

$$(x > 0; \Re(\omega) \geq 0; k = [\Re(\omega)] + 1);$$

Now we recall fractional integral and differential operator of arbitrary order containing Appell function F_3 of first kind in its kernel studed by Saigo and Meada [22]. This operator is more in a general and all other operators behave like a special case of it.

Let $\omega, \omega^{/}, \epsilon, \epsilon^{/}, \eta \in C, x > 0$ then general fractional operators involving Appell function F_3 are defined as

$$\left(I_{0+}^{\omega,\omega^{/},\epsilon,\epsilon^{/},\eta} f\right)(x) = \frac{x^{-\omega}}{\Gamma(\eta)} \int_0^x t^{-\omega} (x-t)^{\eta-1} F_3 \left(\omega, \omega^{/}, \epsilon, \epsilon^{/}; \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t)dt \tag{3.0.39}$$

$$(\Re(\eta) > 0);$$

$$= \frac{d^k}{dx^k} \left(I_{0+}^{\omega,\omega^{/},\epsilon+k,\epsilon^{/},\eta+k} f\right)(x), \quad (\Re(\eta) \leq 0; k = [-\Re(\eta) + 1]);$$

$$\left(I_{0-}^{\omega,\omega^{/},\epsilon,\epsilon^{/},\eta} f\right)(x) = \frac{x^{-\omega^{/}}}{\Gamma(\eta)} \int_0^x t^{-\omega} (t-x)^{\eta-1} F_3 \left(\omega, \omega^{/}, \epsilon, \epsilon^{/}; \eta; 1 - \frac{x}{t}, 1 - \frac{t}{x}\right) f(t)dt \tag{3.0.40}$$

$$(\Re(\eta) > 0);$$

$$= (-1)^k \frac{d^k}{dx^k}\left(I_{0-}^{\omega,\omega^/,\epsilon,\epsilon^/+k,\eta+k} f\right)(x)$$

$$(\Re(\eta) \leq 0; k = [-\Re(\eta) + 1];$$

and

$$\left(D_{0+}^{\omega,\omega^/,\epsilon,\epsilon^/,\eta} f\right)(x)) = \left(I_{0+}^{-\omega^/,-\omega,-\epsilon^/,-\epsilon,-\eta} f\right)(x)) \tag{3.0.41}$$

$$= \frac{d^k}{dx^k}\left(I_{0+}^{-\omega^/,\omega,-\epsilon^/+k,-\epsilon,-\eta+k} f\right)(x)$$

$$(\Re(\eta) > 0; k = [\Re(\eta) + 1])\,;$$

$$(D_{0-}^{\omega,\omega^/,\epsilon,\epsilon^/,\eta} f)(x) = (I_{0-}^{-\omega^/,-\omega,-\epsilon^/,-\epsilon,-\eta} f)(x) \tag{3.0.42}$$

$$= (-1)^k \frac{d^k}{dx^k}(I_{0-}^{-\omega^/,-\omega,-\epsilon^/,-\epsilon+k,-\eta+k} f)(x)$$

$$(\Re(\eta) > 0; k = [\Re(\eta) + 1])\,;$$

Remark 3.0.4. i) Two-variable hypergeometric function known as Appell's function F_3 (see, for details, Rainville [20])and is defined as

$$F_3(\omega,\omega^/,\epsilon,\epsilon^/;\eta;x,y) = \sum_{k=0}^{\infty} \frac{(\omega)_k(\omega^/)_k(\epsilon)_k(\epsilon^/)_k}{(\eta)_{k+l}} \frac{x^k}{k!}\frac{y^k}{l!} \quad (\max\{|x|\,,|y|\} < 1)$$

Appell function F_3 reduces to the Gauss hypergeometric function ${}_2F_1$.
ii) These Marichev-Saigo Maeda operators (3.0.39),(3.0.40)and (3.0.41),(3.0.42) reduces to Saigo operators (3.0.14),(3.0.27) and (3.0.16),(3.0.29) as following.

$$\left(I_{0+}^{\omega,0,\epsilon,\epsilon^/,\eta} f\right)(x) = \left(I_{0+}^{\eta,\omega-\eta,-\epsilon} f\right)(x) \quad (\eta \in C); \tag{3.0.43}$$

$$\left(I_{0-}^{\omega,0,\epsilon,\epsilon^/,\eta} f\right)(x) = \left(I_{0-}^{\eta,\omega-\eta,-\epsilon} f\right)(x) \quad (\eta \in C); \tag{3.0.44}$$

$$\left(D_{0+}^{0,\omega^/,\epsilon,\epsilon^/,\eta} f\right)(x) = \left(D_{0+}^{\eta,\omega^/-\eta,-\epsilon^/-\eta} f\right)(x) \quad (\Re(\eta) > 0); \tag{3.0.45}$$

$$\left(D_{0-}^{0,\omega^/,\epsilon,\epsilon^/,\eta} f\right)(x) = \left(D_{0-}^{\eta,\omega^/-\eta,\epsilon^/-\eta} f\right)(x) \quad (\Re(\eta) > 0); \tag{3.0.46}$$

3.1. **Preliminary Lemmas:** Following Lemmas are essential for establishment of our main results in sequel. These preliminary assertions give composition formulas of fractional integral and fractional differential with a power function.

Lemma 3.1.1. *For fractional integrals and differentiation composition formula with a power function established by Saigo and Meada* [22] *is given bellow.*
If $\omega,\omega^/,\epsilon,\epsilon^/,\eta,\rho \in C, x > 0$ *are such that* $\Re(\omega) > 0, \Re(\rho) > 0\max[0,\Re(\epsilon-\eta)]$. *Then hypergeometric fractional integral formula holds true.*

$$\left(I_{0+}^{\omega,\omega^/,\epsilon,\epsilon^/,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\eta-\omega-\omega^/-\epsilon)\Gamma(\rho+\epsilon^/-\omega^/)}{\Gamma(\rho+\epsilon^/)\Gamma(\rho-\omega-\omega^/+\eta)\Gamma(\rho+\eta-\omega^/-\epsilon)} \tag{3.1.1}$$

$$\times x^{\rho-\omega-\omega^/+\eta-1}$$

$$(x > 0, \Re(\eta) > 0, \Re(\rho) > \max[0, \Re(\omega+\omega^/+\epsilon-\eta), \Re(\omega^/-\epsilon^/)];$$

Thus, the corresponding image formula for Saigo's operator [21].

$$\left(I_{0+}^{\omega,\epsilon,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\eta-\epsilon)}{\Gamma(\rho-\epsilon)\Gamma(\rho+\omega+\eta)} x^{\rho-\epsilon-1}, \quad (x>0); \tag{3.1.2}$$

$$(\Re(\rho) > \max[0, \Re(\epsilon-\eta)])$$

In particular cases, for $\epsilon = -\omega$ *and* $\epsilon = 0$ *respectively, we have*

$$\left(R_{0+}^{\omega} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)}{\Gamma(\rho+\omega)} x^{\rho+\omega-1}, \quad (\Re(\omega) > 0, \Re(\rho) > 0); \tag{3.1.3}$$

$$\left(EK_{0+}^{\omega,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\omega+\eta)} x^{\rho-1}, \quad (\Re(\omega) > 0, \Re(\rho) > -\Re(\eta)); \tag{3.1.4}$$

Corresponding power formula for derivative is as follows

$$\left(D_{0+}^{\omega,\omega^/,\epsilon,\epsilon^/,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma\rho)\Gamma(\rho-\eta+\omega+\omega^/+\epsilon^/)}{\Gamma(\rho-\epsilon)\Gamma(\rho-\eta+\omega+\omega^/)\Gamma(\rho-\eta+\omega+\epsilon^/} x^{\rho-\eta+\omega+\omega^/-1} \tag{3.1.5}$$

$$(\Re(\rho) > \max[0, \Re(\eta-\omega-\omega^/-\epsilon^/), \Re(\epsilon-\omega)]);$$

In continuation to this study, if $\omega, \epsilon, \eta, \rho \in C$ *are such that* $\Re(\omega) > 0, \Re(\rho) > -\min[\Re(\omega), \Re(\eta)]$. *Then hypergeometric fractional derivative formula holds true.*

$$\left(D_{0+}^{\omega,\epsilon,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\omega+\epsilon+\eta)}{\Gamma(\rho+\epsilon)\Gamma(\rho+\eta)} x^{\rho+\epsilon-1} \tag{3.1.6}$$

$$(x > 0; \Re(\omega) \geq 0; \Re(\rho) > -\min[0, \Re(\omega+\epsilon+\eta)]);$$

In particular, if $\epsilon = -\omega$ *and* $\epsilon = 0$ *respectively then we have*

$$({}_{RL}D_{0+}^{\omega} t^{\rho-1})(x) = \frac{\Gamma(\rho)}{\Gamma(\rho-\omega)} x^{\rho-\omega-1}, \quad (\Re(\rho) > \Re(\omega) > 0); \tag{3.1.7}$$

$$({}_{EK}D_{0+}^{\omega,\eta} t^{\rho-1})(x) = \frac{\Gamma(\rho+\omega+\eta)}{\Gamma(\rho+\eta)} x^{\rho-1}, \quad (\Re(\rho) > -\Re(\eta+\omega)); \tag{3.1.8}$$

Lemma 3.1.2. *(* [9], [22]*) If* $\omega, \omega^/, \epsilon, \epsilon^/, \eta, \rho \in C$ *are such that* $\Re(\eta) > 0, \Re(\rho) < 1 + \min[\Re(-\epsilon), \Re(\omega+\omega^/-\eta), \Re(\omega+\epsilon-\eta)]$. *Then fractional integral formula holds true.*

$$\left(I_{0+}^{\omega,\omega^/,\epsilon,\epsilon^/,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(1+\omega+\omega^/-\eta-\rho)\Gamma(1+\omega+\epsilon^/-\eta-\rho)\Gamma(1-\epsilon-\rho)}{\Gamma(1-\rho)\Gamma(1-\rho+\omega+\omega^/+\epsilon^/-\eta)\Gamma(1+\omega-\epsilon-\rho)} \tag{3.1.9}$$

$$\times x^{\rho-\omega-\omega^/+\eta-1}$$

$$(x > 0, \Re(\eta) > 0, 0 < \Re(\rho) < 1 + \min[\Re(-\epsilon), \Re(\omega+\omega^/-\eta), \Re(\omega+\epsilon^/-\eta)];$$

Thus, the corresponding image for Saigo operator

$$\left(I_{0-}^{\omega,\epsilon,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(1+\epsilon-\rho)\Gamma(1+\eta-\rho)}{\Gamma(1-\rho)\Gamma(1-\rho+\omega+\epsilon+\eta)} x^{\rho-\epsilon-1} \quad (x>0); \tag{3.1.10}$$

$$(\Re(\rho) < 1 + \min[\Re(\epsilon), \Re(\eta)]);$$

In particular, if $\epsilon = -\omega$ and $\epsilon = 0$ respectively, then we have

$$(W_{0-}^{\omega} t^{\rho-1})(x) = \frac{\Gamma(1+\omega-\rho)}{\Gamma(1-\rho)} x^{\rho+\omega-1} \quad (1-\Re(\rho) > \Re(\omega) > 0); \tag{3.1.11}$$

$$(EK_{0-}^{\omega,\eta} t^{\rho-1})(x) = \frac{\Gamma(1+\eta-\rho)}{\Gamma(1-\rho+\omega+\eta)} x^{\rho-1} \quad (\Re(\rho) < 1+\Re(\eta)); \tag{3.1.12}$$

Power formula for right-sided Marichev-Saigo-Maeda operator of derivative is as follows

$$\left(D_{0-}^{\omega,\omega^{/},\epsilon,\epsilon^{/},\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho+\epsilon^{/})\Gamma(1-\rho+\eta-\omega-\omega^{/})\Gamma(1-\rho+\eta-\omega^{/}-\epsilon)}{\Gamma(1-rho)\Gamma(1-\rho+\eta-\omega-\omega^{/}-\epsilon)\Gamma(1-\rho-\omega^{/}+\epsilon^{/})} \tag{3.1.13}$$

$$\times x^{\rho-\eta+\omega+\omega^{/}-1}$$

$$(\Re(\rho) < 1+\min[\Re(\epsilon^{/}), \Re(\eta-\omega-\omega^{/}), \Re(\eta-\omega^{/}-\epsilon)]);$$

If $\omega, \epsilon, \eta, \rho \in C$ are such that $\Re(\omega) > 0, \Re(\rho) > -\min[\Re(\epsilon), \Re(\eta)]$ Then hypergeometric fractional derivative formula holds true

$$(D_{0-}^{\omega,\epsilon,\eta} t^{\rho-1})(x) = \frac{\Gamma(1-\rho-\epsilon)\Gamma(1-\rho+\omega+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta-\epsilon)} x^{\rho+\omega-1} \tag{3.1.14}$$

$$(x > 0; \Re(\omega) \geq 0; \Re(\rho) < 1+\min[-\Re(\epsilon+\eta), \Re(\omega+\eta)])$$

In particular, if $\epsilon = -\omega$ and $\epsilon = 0$ respectively, then we have

$$({}_W D_{0-}^{\omega} t^{\rho-1})(x) = \frac{\Gamma(1-\rho+\omega)}{\Gamma(1-\rho)} x^{\rho-\omega-1} \tag{3.1.15}$$

$$({}_{EK} D_{0-}^{\omega,\eta}) t^{\rho-1}(x) = \frac{\Gamma(1-\rho+\omega+\eta)}{\Gamma(1-\rho+\eta)} x^{\rho-1} \tag{3.1.16}$$

Now we give the statement and prove our main fractional integral and differential formulas involving generalized extended Mittag-Leffler function.

4. Left-Sided Generalized Fractional Integral and Differentiation of Extended Mittag-Leffler function

Theorem 4.0.3. *Let $\omega, \omega^{/}, \epsilon, \epsilon^{/}, \eta, \rho \in C$ are complex numbers such that $\Re(\eta) > 0, \Re(\rho) > 0$. Then the fractional calculus of integration and differentiation of extended hypergeometric function holds true. If $\Re(\rho) > \max[0, \Re(\omega+\omega^{/}+\epsilon-\eta), \Re(\omega^{/}-\epsilon^{/})]$ is satisfied then*

$$\left(I_{0+}^{\omega,\omega^{/},\epsilon,\epsilon^{/},\eta}\left(t^{\rho-1} E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(xt^{\sigma};p)\right)\right)(z) = z^{\rho,+\eta-\omega-\omega^{/}-1} \tag{4.0.17}$$

$$E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(xz^{\sigma};p) * {}_4\Psi_3 \left[\begin{array}{c} \Delta \\ \Delta^{/} \end{array} ; xz^{\sigma}\right]$$

where $\Delta = (\rho,\sigma), (\rho+\eta-\omega-\omega^{/}-\epsilon,\sigma), (\rho+\epsilon^{/}-\omega^{/},\sigma), (1,1)$
and $\Delta^{/} = (\rho+\eta-\omega-\omega^{/},\sigma), (\rho+\eta-\omega^{/}-\epsilon,\sigma), (\rho+\epsilon^{/},\sigma)$
and if $\Re(\eta) > 0, \Re(\rho) > \max[0, \Re(\omega+\omega^{/}+\epsilon-\eta), \Re(\omega^{/}-\epsilon^{/})]$ is satisfied then

$$\left(D_{0+}^{\omega,\omega^{/},\epsilon,\epsilon^{/},\eta}\left(t^{\rho-1} E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(xt^{\sigma};p)\right)\right)(z) = z^{\rho-\eta+\omega+\omega^{/}-1} \tag{4.0.18}$$

$$E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(xz^{\sigma};p) * {}_4\Psi_3 \left[\begin{array}{c} \varrho \\ \varrho^{/} \end{array} ; xz^{\sigma}\right]$$

where $\varrho = (\rho,\sigma),(\rho-\eta+\omega+\omega^{/}+\epsilon^{/},\sigma),(\rho-\epsilon+\omega,\sigma),(1,1)$
and $\varrho^{/} = (\rho-\eta+\omega+\omega^{/},\sigma),(\rho-\eta+\omega+\epsilon^{/},\sigma),(\rho-\epsilon,\sigma)$
It is supposed that left-sided fractional integral (4.0.17) and derivative (4.0.18) of extended Mittag-Leffler function exist.

Proof. For the sake of conveniences, let right-hand side of (4.0.17) is labeled as $\Omega(z)$ and by the extension form of the function (1.0.5), we have after changing the order of integration and summation due to Dirichlet formula by Samko et al. [23].

$$\Omega(z) = \sum_{k=0}^{\infty} \frac{B_p^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)} \frac{x^k}{\Gamma(\xi k+\zeta)} \left(I_{0+}^{\omega,\omega^{/},\epsilon,\epsilon^{/},\eta} t^{\rho+\sigma k-1}\right)(z)$$

Due to absolute and uniform convergence of the series, under specific conditions, integral and summations are justified. Thus, by lemma (3.1.1), we have after replacing ρ by $\rho+\sigma k$

$$\Omega(z) = \sum_{k=0}^{\infty} \frac{B_p^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)} \frac{x^k}{\Gamma(\xi k+\zeta)}$$

$$\times \frac{\Gamma(\rho+\sigma k)\Gamma(\rho+\eta-\omega-\omega^{/}-\epsilon+\sigma k)\Gamma(\rho+\epsilon^{/}-\omega^{/}+\sigma k)}{\Gamma(\rho+\eta-\omega-\omega^{/}+\sigma k)\Gamma(\rho+\eta-\omega-\epsilon+\sigma k)\Gamma(\rho+\epsilon+\sigma k)} z^{\rho-\omega-\omega^{/}+\eta+\sigma k-1}$$

Inter-operating the last member in terms of Hadamard product series, we have

$$\Omega(z) = z^{\rho+\eta-\omega-\omega^{/}-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xz^{\sigma};p) * {}_4\Psi_3\left[\begin{array}{c}\Delta\\ \Delta^{/}\end{array};xz^{\sigma}\right]$$

where $\Delta = \left[(\rho,\sigma),(\rho+\eta-\omega-\omega^{/}-\epsilon,\sigma),(\rho+\epsilon^{/}-\omega^{/},\sigma),(1,1)\right]$
and $\Delta^{/} = \left[(\rho+\eta-\omega-\omega^{/},\sigma),(\rho+\eta-\omega^{/}-\epsilon,\sigma),(\rho+\epsilon^{/},\sigma)\right]$
Hence

$$\left(I_{0+}^{\omega,\omega^{/},\epsilon,\epsilon^{/},\eta}\left(t^{\rho-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(xt^{\sigma};p)\right)\right)(z) = z^{\rho-\eta-\omega-\omega^{/}-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xz^{\sigma};p) * {}_4\Psi_3\left[\begin{array}{c}\Delta\\ \Delta^{/}\end{array};xz^{\sigma}\right]$$

Now, for derivative

$$\left(D_{0+}^{\omega,\omega^{/},\epsilon,\epsilon^{/},\eta}\left(t^{\rho-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(xt^{\sigma};p)\right)\right)(z)$$

$$= \left(\frac{d}{dz}\right)^m \sum_{k=0}^{\infty} \frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}$$

$$\times \frac{x^k}{\Gamma(\xi k+\zeta)} \left(I_{0+}^{-\omega^{/},-\omega,-\epsilon^{/}+m,-\epsilon,-\eta+m} t^{\rho+\sigma k-1}\right)(x)$$

$$= \sum_{k=0}^{\infty} \frac{B_p^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)} \frac{x^k}{\Gamma(\xi k+\zeta)}$$

$$\times \frac{\Gamma(\rho+\sigma k)\Gamma(\rho+\sigma k-\eta+\omega+\omega^{/}+\epsilon^{/})\Gamma(\rho+\sigma k-\epsilon+\omega)}{\Gamma(\rho+\sigma k-\eta+m+\omega+\omega^{/})\Gamma(\rho+\sigma k-\eta+\omega+\epsilon^{/})\Gamma(\rho+\sigma k-\epsilon)}$$

$$\times \left(\frac{d}{dz}\right)^m z^{\rho+\sigma k-\eta+m+\omega+\omega^{/}-1}$$

Now using $\frac{d^m z^k}{dz^m} = \frac{\Gamma(k+1)}{\Gamma(k-m+1)} z^{k-m}, \quad k \geq m$

$$= z^{\rho-\eta+\omega+\omega^{/}-1} \sum_{k=0}^{\infty} \frac{B_p^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{(xz^\sigma)^k}{\Gamma(\xi k+\zeta)}$$

$$\times \frac{\Gamma(\rho+\sigma k)\Gamma(\rho-\eta+\omega+\omega^{/}+\epsilon^{/}+\sigma k)\Gamma(\rho-\epsilon+\omega+\sigma k)}{\Gamma(\rho-\eta+\omega+\omega^{/}+\sigma k)\Gamma(\rho-\eta+\omega+\epsilon^{/}+\sigma k)\Gamma(\rho-\epsilon+\sigma k)}$$

$$= z^{\rho-\eta+\omega+\omega^{/}-1} E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xz^\sigma; p) * {}_4\Psi_3 \left[\begin{array}{c} \varrho \\ \varrho^{/} \end{array} ; xz^\sigma \right]$$

where $\varrho = \left[(\rho, \sigma), (\rho-\eta+\omega+\omega^{/}+\epsilon^{/}, \sigma), (\rho-\epsilon+\omega, \sigma), (1, 1)\right]$
and $\varrho^{/} = \left[(\rho-\eta+\omega+\omega^{/}, \sigma), (\rho-\eta+\omega+\epsilon^{/}, \sigma), (\rho-\epsilon, \sigma)\right]$
Now, the corresponding results for Saigo, Riemann-Liouville and Erdelyi-Kober operators are as follows □

Corollary 4.0.4. *Let $\omega, \epsilon, \eta, \rho \in C$ be such that $\Re(\omega) > 0, \Re(\rho) > max[0, \Re(\epsilon - \eta)]$, then the left-sided Saigo fractional integral formula for extended Mittag-Leffler function holds true.*

$$\left(I_{0+}^{\omega,\epsilon,\eta}\left(t^{\rho-1} E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xt^\sigma; p)\right)\right)(z) = z^{\rho-\epsilon-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xz^\sigma; p) * {}_3\Psi_2 \left[\begin{array}{c} (\rho, \sigma), (\rho-\epsilon+\eta, \sigma), (1, 1) \\ (\rho-\epsilon, \sigma), (\rho+\omega+\eta, \sigma) \end{array} ; xz^\sigma \right]$$

and under the stated conditions, the left-sided Saigo fractional derivative formula for extended Mittag-Leffler function holds true.

$$\left(D_{0+}^{\omega,\epsilon,\eta}\left(t^{\rho-1} E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xt^\sigma; p)\right)\right)(z) = z^{\rho+\epsilon-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xz^\sigma; p) * {}_3\Psi_2 \left[\begin{array}{c} (\rho, \sigma), (\rho+\omega+\epsilon+\eta, \sigma), (1, 1) \\ (\rho+\epsilon, \sigma), (\rho+\eta, \sigma) \end{array} ; xz^\sigma \right]$$

Corollary 4.0.5. *Let $\omega, \epsilon, \eta, \rho \in C$ be such that $\Re(\omega) > 0, \Re(\rho) > 0, |x| < 1$, then the left-sided Riemann Liouville fractional integral formula for extended Mittag-Leffler function holds true.*

$$\left(R_{0+}^{\omega}\left(t^{\rho-1} E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xt^\sigma; p)\right)\right)(z) = z^{\rho+\omega-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xz^\sigma; p) * {}_2\Psi_1 \left[\begin{array}{c} (\rho, \sigma), (1, 1) \\ (\rho+\omega, \sigma) \end{array} ; xz^\sigma \right]$$

and under the stated conditions, the left-sided Riemann Liouville fractional derivative formula for extended Mittag-Leffler function holds true.

$$\left({}_{RL}D_{0+}^{\omega}\left(t^{\rho-1} E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(xt^\sigma; p)\right)\right)(z) = z^{\rho-\omega-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xz^\sigma; p) * {}_2\Psi_1 \left[\begin{array}{c} (\rho, \sigma), (1, 1) \\ (\rho-\omega, \sigma) \end{array} ; xz^\sigma \right]$$

In continuation of corollaries of operators, corresponding result for Erdelyi-Kober fractional operators of left-sided integration and differentiation are as follows.

Corollary 4.0.6. *Let $\omega, \epsilon, \eta, \rho \in C$ be such that $\Re(\omega) > 0, \Re(\rho) > 0$ then the left-sided Erdelyi-Kober fractional integral formula for extended Mittag-Leffler function holds true.*

$$\left(K_{0+}^{\omega,\eta}\left(t^{\rho-1}E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xt^{\sigma};p)\right)\right)(z) = z^{\rho-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xz^{\sigma};p) * {}_2\Psi_1\left[\begin{array}{c}(\rho+\eta,\sigma),(1,1)\\(\rho+\omega+\eta,\sigma)\end{array};xz^{\sigma}\right]$$

and under the stated conditions, the left-sided Erdelyi-Kober fractional derivative formula for extended Mittag-Leffler function holds true.

$$\left({}_{EK}D_{0+}^{\omega,\eta}\left(t^{\rho-1}E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xt^{\sigma};p)\right)\right)(z) = z^{\rho-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(xz^{\sigma};p) * {}_2\Psi_1\left[\begin{array}{c}(\rho+\omega+\eta,\sigma),(1,1)\\(\rho+\eta,\sigma)\end{array};xz^{\sigma}\right]$$

5. Right-side Generalized Fractional Integration and Differentiation of Extended Mittag-Leffler Function

Theorem 5.0.7. *Let $\omega, \omega^{/}, \epsilon, \epsilon^{/}, \eta, \rho \in C$ be a complex number such that $\Re(\eta) > 0, \Re(\rho) > 0$. Then the right-sided fractional calculus of integration and differentiation of extended Mittag-Leffler function holds true under the condition $\Re(\rho) < 1 + \min[\Re(-\epsilon), \Re(\omega+\omega^{/}-\eta), \Re(\omega+\epsilon^{/}-\eta)]$*

$$\left(I_{0-}^{\omega,\omega^{/},\epsilon,\epsilon^{/},\eta}\left(t^{\rho-1}E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(\frac{x}{t^{\sigma}};p)\right)\right)(z) = z^{\rho+\eta-\omega-\omega^{/}-1} \tag{5.0.19}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}\left(\frac{x}{z^{\sigma}};p\right) * {}_4\Psi_3\left[\begin{array}{c}\tau\\\tau^{/}\end{array};\frac{x}{z^{\sigma}}\right]$$

where $\tau = [(1-\rho-\epsilon,\sigma),(1-\rho-\eta+\epsilon^{/}+\omega,\sigma),(1-\rho-\eta+\omega+\omega^{/},\sigma)(1,1)]$
and $\tau^{/} = [(1-\rho,\sigma),(1-\rho+\omega-\epsilon,\sigma),(1-\rho+\omega+\omega^{/}+\epsilon-\eta,\sigma)]$
and

$$\left(D_{0-}^{\omega,\omega^{/},\epsilon,\epsilon^{/},\eta}\left(t^{\rho-1}E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(\frac{x}{t^{\sigma}};p)\right)\right)(z) = z^{\rho+\eta-\omega-\omega^{/}-1} \tag{5.0.20}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}\left(\frac{x}{z^{\sigma}};p\right) * {}_4\Psi_3\left[\begin{array}{c}\chi\\\chi^{/}\end{array};\frac{x}{z^{\sigma}}\right]$$

where $\chi = [(1-\rho+\eta-\omega-\omega^{/},\sigma),(1-\rho+\epsilon^{/},\sigma),(1-\rho+\eta-\omega^{/}-\epsilon,\sigma)(1,1)]$
and$\chi^{/} = [(1-\rho,\sigma),(1-\rho+\epsilon^{/}-\omega^{/},\sigma),(1-\rho-\omega-\omega^{/}-\epsilon+\eta,\sigma)]$
It is supposed that right-sided fractional integral (5.0.19) and derivative (5.0.20) of extended Mittag-Leffler function exist.

Proof. Let right-hand side of (5.0.19) be labeled as $\Lambda(z)$ and by the definition of extended function (1.0.5), we have after changing the order of integration and summation due to Dirichlet formula (see [23]).

$$\Lambda(z) = \sum_{k=0}^{\infty}\frac{B_p^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\xi k+\zeta)}\left(I_{0-}^{\omega,\omega^{/},\epsilon,\epsilon^{/},\eta}(t^{\rho-\sigma k-1})\right)(z)$$

Due to absolute and uniform convergence of the series, under specific conditions, integral and summations are justified. Thus, by lemma(3.1.2), we have after replacing ρ by $\rho-\sigma k$

$$\Lambda(z)=\sum_{k=0}^{\infty}\frac{B_p^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\xi k+\zeta)}$$

$$\times\frac{\Gamma(1-\rho-\eta+\omega+\omega^/+\sigma k)\Gamma(1-\rho-\eta+\omega+\epsilon^/+\sigma k)\Gamma(1-\rho-\epsilon+\sigma k)}{\Gamma(1-\rho+\sigma k)\Gamma(1-\rho+\omega+\omega^/+\epsilon^/-\eta+\sigma k)\Gamma(1-\rho+\omega-\epsilon+\sigma k)}$$

$$\times z^{\rho-\omega-\omega^/+\eta-\sigma k-1}$$

Expressing the last term by Hadamard product series, we have

$$\Lambda(z)=z^{\rho-\omega-\omega^/+\eta-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}\left(\frac{x}{z^\sigma};p\right)*{}_4\Psi_3\left[\begin{array}{c}\tau\\ \tau^/\end{array};\frac{x}{z^\sigma}\right]$$

Since, the proof of the fractional differential formula would run parallel to the proof of fractional integral formula (5.0.20). We, therefore, skip the details involved. □

Corollary 5.0.8. *Let $\omega,\epsilon,\eta,\rho\in C$ be a complex number such that $\Re(\eta)>0,\Re(\rho)>0$. Then the right-sided fractional calculus of integration and differentiation of extended Mittag-Leffler function holds true under the condition $\Re(\rho)<1+\min[\Re(-\epsilon),\Re(\omega+\omega^/-\eta),\Re(\omega+\epsilon^/-\eta)]$*

$$\left(I_{0-}^{\omega,\epsilon,\eta}\left(t^{\rho-1}E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}\left(\frac{x}{t^\sigma};p\right)\right)\right)(z)=z^{\rho-\epsilon-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}\left(\frac{x}{z^\sigma};p\right)*{}_3\Psi_2\left[\begin{array}{c}(1-\rho+\epsilon,\sigma),(1-\rho+\eta,\sigma),(1,1)\\ (1-\rho,\sigma),(1-\rho+\omega+\epsilon+\eta,\sigma)\end{array};\frac{x}{z^\sigma}\right]$$

and under the condition $[\Re(\rho)>\min[\Re(-\epsilon-k),\Re(\omega+\eta)]$ and $k=[\Re(\omega)+1]$

$$\left(D_{0-}^{\omega,\epsilon,\eta}\left(t^{\rho-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\left(\frac{x}{t^\sigma};p\right)\right)\right)(z)=z^{\rho+\epsilon-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}\left(\frac{x}{z^\sigma};p\right)*{}_3\Psi_2\left[\begin{array}{c}(1-\rho-\epsilon,\sigma),(1-\rho+\omega+\eta,\sigma),(1,1)\\ (1-\rho,\sigma),(1-\rho+\eta-\epsilon,\sigma)\end{array};\frac{x}{z^\sigma}\right]$$

Further, the corresponding result for right-sided Weyl and Erdelyi-Kober fractional operators following the Marichev-Saigo-Maeda operators are as follows.

Corollary 5.0.9. *Let $\omega,\rho\in C$ be such that $\Re(\omega)>0,\Re(\rho)>0$. Then the above result reduces to the following result.*

$$\left(W_{0-}^{\omega}\left(t^{\rho-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\left(\frac{x}{t^\sigma};p\right)\right)\right)(z)=z^{\rho+\omega-1}$$

$$E_{\xi,\zeta}^{\left(\{\kappa_n\}_{n\in N_0};\gamma\right)}\left(\frac{x}{z^\sigma};p\right)*{}_2\Psi_1\left[\begin{array}{c}(1-\rho-\omega,\sigma),(1,1)\\ (1-\rho,\sigma)\end{array};\frac{x}{z^\sigma}\right]$$

and under the stated condition, the following Weyl derivative formula holds true.

$$\left({}_WD_{0-}^{\omega}\left(t^{\rho-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\left(\frac{x}{t^\sigma};p\right)\right)\right)(z)=z^{\rho-\omega-1}$$

$$E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\left(\frac{x}{z^\sigma};p\right)*{}_2\Psi_1\left[\begin{array}{c}(1-\rho+\omega,\sigma),(1,1)\\ (1-\rho,\sigma)\end{array};\frac{x}{z^\sigma}\right]$$

Corollary 5.0.10. *Let $\omega, \eta, \rho \in C$ be such that $\Re(\omega) > 0, \Re(\rho) > 0$. Then the above result reduces to the following result of Erdelyi-Kober formula of integration and differentiation.*

$$\left(K_{0-}^{\omega,\eta}\left(t^{\rho-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\frac{x}{t^{\sigma}};p)\right)\right)(z) = z^{\rho-1}$$

$$E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\left(\frac{x}{z^{\sigma}};p\right) * {}_2\Psi_1\left[\begin{array}{c}(1-\rho+\eta,\sigma),(1,1)\\(1-\rho+\omega+\eta,\sigma)\end{array};\frac{x}{z^{\sigma}}\right]$$

and.

$$({}_{EK}D_{0-}^{\omega}(t^{\rho-1}E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}(\frac{x}{t^{\sigma}};p)))(z) = z^{\rho-1}$$

$$E_{\xi,\zeta}^{(\{\kappa_n\}_{n\in N_0};\gamma)}\left(\frac{x}{z^{\sigma}};p\right) * {}_2\Psi_1\left[\begin{array}{c}(1-\rho+\omega+\eta,\sigma),(1,1)\\(1-\rho+\eta,\sigma)\end{array};\frac{x}{z^{\sigma}}\right]$$

Conclusion Remakes

In this investigation, we obtained composite formulas of fractional integration (3.0.39),(3.0.40) and differentiation (3.0.41),(3.0.42) known as Marichev-Saigo-Maedea operators with the extension of Mittag-Leffler function (1.0.5) with the help of Hadamard product (2.0.12). Hadamard product provides a chance to get the solution of such complicated functions in terms of two known analytical functions. Further, our main theorems of left and right-sided of integration and differentiation are helpful to reduce the results of other known operators like Saigo, Riemann-Liouville, Weyl and Erdelyi-Kober operators.

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