Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 10 Issue 1(2019), Pages 161-166.

WILKER AND HUYGENS-TYPE INEQUALITIES FOR SOME BIVARIATE MEANS WITH APPLICATIONS TO ELEMENTARY FUNCTIONS

EDWARD NEUMAN

Abstract. The goal of this paper is to establish Wilker and Huygens-type inequalities involving ratios of some bivariate means. The means in question are the power means, the identric mean and the power-exponential mean. Corollaries of the main results involve Wilker and-Huygens type inequalities involving hyperbolic functions and the natural exponential function.

1. Introduction

In recent years certain bivariate means have been investigated extensively by several researchers. A complete list of research papers which deal with this subject is too long to be included here even if we would restrict our attention to papers published in the last ten years. The goal of this paper is to obtain Wilker and Huygens-type inequalities involving bivariate means such as the power-exponential mean, identric mean and unweighted power mean. The first mean mentioned here has been studied by several researchers (see, e.g., [18, 19, 20, 13]).

Let u, v, α and β be positive numbers. The following inequality

$$\alpha + \beta < \alpha u^p + \beta v^q \tag{1.1}$$

is often called the Wilker and Huygens-type inequality. Inequalities such as (1.1) have been established when u and v involve either circular, hyperbolic, lemniscate functions, generalized Jacobian elliptic, generalized trigonometric and generalized hyperbolic functions and also for the theta functions. The interested reader is referred to [1, 14, 2, 3, 4, 5, 6, 8, 9, 10, 15, 16, 17].

2. Some Bivariate Means Used in This Paper

In what follows the letters a and b will always stand for positive and unequal numbers.

2000 *Mathematics Subject Classification.* 35A07, 35Q53.
Key words and phrases. Wilker and Huygens-type inequalities, the power-exponential mean, identric mean, power mean.

Submitted October 16, 2018. Published January 25, 2019.
Communicated by Valmir Krasniqi.

For the reader's convenience we recall first definition of the power-exponential mean of a and b:

$$Z(a,b) \equiv Z = (a^a b^b)^{\frac{1}{a+b}}. \tag{2.1}$$

The unweighted power mean of order r of a and b is given by

$$A_r(a,b) \equiv A_r = \begin{cases} \sqrt[r]{\dfrac{a^r+b^r}{2}} & \text{if } \ r \neq 0, \\ \sqrt{ab} & \text{if } \ r = 0. \end{cases} \tag{2.2}$$

Throughout the sequel we will adopt common notation for the power means of order 0, 1, and 2. These are denoted by G, A and Q, respectively. The identric mean of a and b is defined as follows

$$I(a,b) \equiv I = e^{-1}(\frac{a^a}{b^b})^{\frac{1}{a-b}}. \tag{2.3}$$

All the means mentioned in this section can be represented in terms of the variable

$$x = \frac{1}{2}\ln(a/b). \tag{2.4}$$

It is known that

$$Z(a,b) = G(a,b)e^{x \tanh x}. \tag{2.5}$$

Similarly,

$$A_r(a,b) = G(a,b)(\cosh rx)^{\frac{1}{r}}, \tag{2.6}$$

and

$$I(a,b) = G(a,b)e^{x \coth x - 1}. \tag{2.7}$$

For the later use let us record a chain of inequalities mentioned in this section

$$G < I < A_{\ln 2} < A_2 < Z \tag{2.8}$$

(see, e.g.[11, 12]).

3. A preliminary result

Proofs of the main results of this paper (see Section 4) require application of the following (see [7]):

Theorem 3.1. *Let u, v, α, β, γ and δ be positive numbers which satisfy the following conditions*
(i) $min(u,v) < 1 < max(u,v)$,
(ii) $1 < u^\gamma v^\delta$.
Then the inequality (1.1) is satisfied provided $u < 1 < v$ and

$$q > 0 \qquad \text{and} \qquad p\alpha\delta \leq q\beta\gamma. \tag{3.1}$$

If in addition the positive numbers u, v, γ and δ are such that
(iii) $\gamma + \delta < \gamma\dfrac{1}{u} + \delta\dfrac{1}{v}$,
then the inequality (1.1) is satisfied if

$$p \leq q \leq -1 \qquad \text{and} \qquad \beta\gamma \leq \alpha\delta. \tag{3.2}$$

Conditions of validity of (1.1) when $v < 1 < u$ are also obtained in [7]. The counterpart of (3.1) is

$$p > 0 \qquad \text{and} \qquad q\beta\gamma \leq p\alpha\delta \tag{3.3}$$

while the conditions (3.2) are replaced now by

$$q \leq p \leq -1 \qquad \text{and} \qquad \alpha\delta \leq \beta\gamma. \tag{3.4}$$

Theorem 3.1 can be employed when deriving Wilker and Huygens-type inequalities for the ratios of means of several variables. Suppose that X, Y and Z are multivariable means which are comparable, i.e., let $X < Y < Z$. With

$$u = \frac{Y}{Z} \qquad \text{and} \qquad v = \frac{Y}{X}$$

we see that the separation condition $u < 1 < v$ is satisfied. In what follows we will always assume that the numbers γ and δ are nonnegative whose sum is equal to 1. If the means in question are such that

$$X^{\gamma} Z^{\delta} < Y,$$

then clearly the last inequality is the same as the inequality of condition (ii) which is utilized in the proof of conditions (3.1). See, [7]. If in addition the means in question satisfy the inequality

$$Y < \gamma X + \delta Z,$$

then this relation is equivalent to the inequality of condition (iii) which in turn imply conditions (3.2).

4. Main Results

The first theorem of this paper reads as follows

Theorem 4.1. *Let α and β be positive numbers. Then the inequality*

$$\alpha + \beta < \alpha(\frac{A}{Z})^p + \beta(\frac{A}{G})^q \tag{4.1}$$

holds true if either

$$q > 0 \qquad \textit{and} \qquad p\alpha \leq q\beta \tag{4.2}$$

or if

$$p \leq q \leq -1 \qquad \textit{and} \qquad \beta \leq \alpha. \tag{4.3}$$

Proof. We let

$$u = \frac{A}{Z} \qquad \text{and} \qquad v = \frac{A}{G}.$$

Making use of the inequality

$$G < A < Z$$

we see that u and v satisfy the separation condition $u < 1 < v$. In order to establish the assertion we shall employ the following two-sided inequality (cf. [18, 19]):

$$(ZG)^{\frac{1}{2}} < A < \frac{Z+G}{2}. \tag{4.4}$$

Utilizing the two-sided inequality (4.4) and the formulas for u and v we obtain

$$1 < u^{\frac{1}{2}} v^{\frac{1}{2}} \tag{4.5}$$

and

$$1 < \frac{1}{2}(\frac{1}{u} + \frac{1}{v}) \tag{4.6}$$

We see that u and v satisfy conditions (i)-(iii) of Theorem 3.1 with $\gamma = \delta = 1/2$. Inequalities (4.1) through (4.3) follow now from (1.1), (3.1), and (3.2). The proof is complete. □

A new Wilker and Huygens-type inequality involving hyperbolic and natural exponential function follows from Theorem 4.1. We have

Corollary 4.2. *If the numbers p, q, α and β satisfy conditions (4.2) and (4.3), then for all nonzero numbers x one has*

$$\alpha + \beta < \alpha\Big(\frac{\cosh\, x}{e^{x \tanh\, x}}\Big)^p + \beta(\cosh\, x)^q. \tag{4.7}$$

To obtain inequality (4.7) it suffices to apply formulas (2.5) and (2.6) to (4.1).

Our next result reads as follows

Theorem 4.3. *Let α and β be positive numbers. Then the inequality*

$$\alpha + \beta < \alpha(\frac{A}{Z})^p + \beta(\frac{A}{I})^q \tag{4.8}$$

holds true if

$$q > 0 \qquad \textit{and} \qquad 3p\alpha \le q\beta. \tag{4.9}$$

Proof. Let now

$$u = \frac{A}{Z} \qquad \text{and} \qquad v = \frac{A}{I}.$$

The well known inequality $I < A < Z$ yields the separation condition

$$u < 1 < v.$$

Now we appeal to J. Sándor's result (cf. [13])

$$ZI^3 < A^4$$

to obtain

$$1 < u^{\frac{1}{4}} v^{\frac{3}{4}}.$$

The assumptions $(i) - (iii)$ are satisfied with $\gamma = 1/4$ and $\delta = 3/4$. We again employ Theorem 3.1 to obtain the asserted result. □

Another Wilker and Huygens-type inequality involving hyperbolic functions and the exponential function can be derived using inequality for means. We have

Corollary 4.4. *If the numbers p, q, α and β satisfy conditions (4.9), then for all nonzero numbers x one has*

$$\alpha + \beta < \alpha\Big(\frac{\cosh\, x}{e^{x \tanh\, x}}\Big)^p + \beta\Big(\frac{\cosh\, x}{e^{x \coth x - 1}}\Big)^q. \tag{4.10}$$

The assertion follows from Theorem 4.3 with the use of formulas (2.5), (2.6) and (2.7).

Before we will state and prove the last result of this note let us introduce more notation. Let

$$u = \frac{I}{A_r}, \qquad v = \frac{Z}{A_r}, \tag{4.11}$$

where $\ln 2 < r < 2$. We have the following

Theorem 4.5. *Let α and β be positive numbers and let p and q satisfy the following conditions*

$$q > 0 \qquad \text{and} \qquad p\alpha \leq q\beta. \tag{4.12}$$

Then the inequality

$$\alpha + \beta < \alpha(\frac{I}{A_r})^p + \beta(\frac{Z}{A_r})^q \tag{4.13}$$

holds true.

Proof. The asserted result is obtained with the aid of the following two-sided inequality

$$I < A_r < Z \tag{4.14}$$

(see (2.8)). This in conjunction with (4.11) yields

$$u < 1 < v.$$

Also, the inequality

$$A_r < (ZI)^{\frac{1}{2}}$$

(see the first inequality of (3.11) in [8]) yields

$$1 < u^{\frac{1}{2}} v^{\frac{1}{2}}.$$

Thus the conditions (i)-(iii) of Theorem 3.1 are satisfied with $\gamma = \delta = 1/2$. We appeal again to Theorem 3.1 to obtain the desired result. The proof is complete. □

A corollary, which is a consequence of the last theorem, reads as follows

Corollary 4.6. *Let the positive numbers α, β be such that $\alpha \leq \beta$. Then the inequality*

$$(\alpha + \beta)(\cosh rx)^{\frac{p}{r}} < \alpha e^{p(x \coth x - 1)} + \beta e^{px \tanh x} \tag{4.15}$$

holds true for all $x \neq 0$, for all $p > 0$, and for all numbers r such that $\ln 2 < r < 2$.

Proof. We let $q = p$ and next utilize formulas (2.5)-(2.7) together with Theorem 4.5, to obtain the asserted result. □

References

[1] C. Huygens, *Oeuvres Completes 1888–1940*, Sociéte Hollondaise des Science, Haga.

[2] E. Neuman, *On lemniscate functions.* Integral Transforms Spec. Funct. **15** (2013), no. 3, pp. 164–171.

[3] E. Neuman, *Wilker and Huygens type inequalities for Jacobian elliptic and theta functions.* Integral Transforms Spec. Funct. **25** (2014), no. 3, pp. 240–248.

[4] E. Neuman, *Inequalities involving generalized Jacobian elliptic functions.* Integral Transforms Spec. Funct. **25** (2014), no. 11, pp. 864–873.

[5] E. Neuman, *Wilker and Huygens type inequalities for the generalized trigonometric and for the generalized hyperbolic functions.* Appl. Math. Comput. **230** (2014), pp. 211–217.

[6] E. Neuman, *Wilker and Huygens type inequalities for some elementary functions and Eulerian numbers.* Adv. Studies Contemp. Math. **25** (2015), no. 2, pp. 189–194.

[7] E. Neuman, *Inequalities for weighted sums of powers and their applications*, Math. Inequal. Appl. **15** (2012) 995–1005.

[8] E. Neuman, J. Sándor, *Inequalities involving Stolarsky and Gini means*, Math. Pannon. **14** (2003) 29–44.

[9] E. Neuman, J. Sándor, *Companion inequalities for certain bivariate means*, Appl. Anal. Discr. Math. **3** (2010) 46–57.

[10] E. Neuman, J. Sándor, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inequalities*, Math. Inequal. Appl. **3** (2010) 715–723.

[11] E. Neuman, J. Sándor, *Optimal inequalities for hyperbolic and trigonometric functions*, Bull. Math. Anal. Appl. **3** (2011) 177–181.
[12] E. Neuman, J. Sándor, *Inequalities for hyperbolic functions* Appl. Math. Appl. (2012) **218** 9291–9295.
[13] J. Sándor, *On certain inequalities for means*, Studia Univ. Babes-Bolyai Math. Cluj **38** (1993) 7–14.
[14] J.B. Wilker, *Problem 3306*, Amer. Math. Monthly **96** (1989), 55.
[15] S.-H. Wu and H.M. Srivastava, *A weighted and exponential generalization of Wilker's inequality and its applications*, Integral Transform. and Spec. Funct. **18** (2007), No. 8, 525–535.
[16] L. Zhu, *A new simple proof of Wilker's inequality*, Math. Inequal. Appl. **8** (2005), No. 4, 749–750.
[17] L. Zhu, *On Wilker-type inequalities*, Math. Inequal. Appl. **10** (2007), No. 4, 727–731.
[18] Zhen-H. Yang, *On the log-convexity of two parameter homogeneous functions*, Math. Inequal Appl. **10** (2007) 499–516.
[19] Zhen-H. Yang, *The log-convexity of another class of one-parameter means and its applications*, Bull. Korean Math. Soc. **49** (2012) 33–47.
[20] Zhen-H. Yang, Jengfeng Tian, *Optimal inequalities involving power-exponential mean, arithmetic mean and geometric mean*, Math. Inequal. Appl. **11** 2 (2017) 1169–1183.

Edward Neuman
Mathematical Research Institute, 144 Hawthorn Hollow, Carbondale, IL 62903 Southern Illinois University, Carbondale, IL 62901, USA
E-mail address: edneuman76@gmial.com