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GENERALIZATION OF KY FAN INEQUALITY AND RELATED RESULTS

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ABSTRACT. The aim of this article is to give refinements, extension and produce some variants and related results of generalized Ky Fan inequality.

1. Preliminaries

Throughout this paper, for all $x_j \in [c,d]$ and $w_j > 0, j \in \{1,\ldots,n\}$ with $\sum_{j\in I} w_j = 1$, where $I \subset \{1,\ldots,n\}$, we denote by A_I, G_I and H_I for arithmetic, geometric and harmonic weighted mean respectively for all $x_j \in [c,d]$, $0 < c < d$. That is,

$$A_I = \left(c + d - \sum_{j\in I} w_j x_j\right) \tag{1.1}$$

$$G_I = \frac{cd}{\prod_{j\in I} x_j^{w_j}} \tag{1.2}$$

$$H_I = \left(\frac{1}{c} + \frac{1}{d} - \sum_{j\in I} w_j \frac{1}{x_j}\right)^{-1} \tag{1.3}$$

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and also A'_I, G'_I and H'_I for arithmetic, geometric and harmonic weighted mean for all $(1-x_j) \in [c,d]$ respectively,

$$A'_I = \left((1-c)+(1-d)-\sum_{j\in I} w_j(1-x_j)\right) \tag{1.4}$$

$$G'_I = \frac{(1-c)(1-d)}{\left(\prod_{j\in I}(1-x_j)^{w_j}\right)} \tag{1.5}$$

$$H'_I = \left(\frac{1}{1-c}+\frac{1}{1-d}-\sum_{j\in I} w_j \frac{1}{(1-x_j)}\right)^{-1} \tag{1.6}$$

further we define $\sum_{j=1}^{p} w_j = W_p = W.$

2. Introduction

In 1961, the following inequality was given in the famous monograph 'Inequalities' by E.F. Beckenbach and R. Bellman in [7] due to Ky Fan, which has attracted the attention of various mathematician and its numerous proof, extensions and refinements has been proved.

$$\frac{G_n}{G'_n} \leq \frac{A_n}{A'_n}, \quad x_j \in (0, \frac{1}{2}] \tag{2.1}$$

equality holds iff $x_1 = \cdots = x_n$.

In 1984, W. L. Wang and P. F. Wang in [14] proved the following inequality for the special case when $p_1 = \cdots = p_n = \frac{1}{n}$, later in 1989, H. Alzer proved this inequality for weighted mean that is,

$$\frac{H_n}{H'_n} \leq \frac{G_n}{G'_n}, \quad x_j \in (0, \frac{1}{2}] \tag{2.2}$$

In 1988, H. Alzer in [5] established an additive analogue of $\frac{A_n}{A'_n}$ and $\frac{G_n}{G'_n}$, that is,

$$G_n - G'_n \leq A_n - A'_n, \quad x_j \in (0, \frac{1}{2}] \tag{2.3}$$

equality holds iff $x_1 = \cdots = x_n$.

In 1990, J. Sandor was discovered in [12] for the first time the following inequality. Later in 2003, J. Rooin proved it in [11].

$$\frac{1}{A_n} - \frac{1}{A'_n} \leq \frac{1}{H_n} - \frac{1}{H'_n}, \quad x_j \in (0, \frac{1}{2}] \tag{2.4}$$

equality holds iff $x_1 = \cdots = x_n$.

In 2003, J. Rooin [11] gave the refinements of Ky Fan inequality for weighted mean that is,

$$\frac{A'_n}{G'_n} \leq \frac{1-G_n}{1-A_n} \leq \frac{A_n}{G_n}, \quad x_j \in (0, \frac{1}{2}] \tag{2.5}$$

$$\frac{A'_I}{G'_n} \leq \frac{1-G'_n}{1-A'_n} \leq \frac{A_n}{G_n}, \quad x_j \in (0, \frac{1}{2}] \tag{2.6}$$

equality holds iff $x_1 = \cdots = x_n$.

In 2008, J. Rooin in [10] proved the Ky Fan inequality by dividing left and right hands of it by $\dfrac{1}{G_n + G'_n}$ that is,

$$\frac{A'_n}{G'_n} \leq \frac{1}{G_n + G'_n} \leq \frac{A_n}{G_n}, \quad x_j \in (0, \frac{1}{2}] \tag{2.7}$$

equality holds iff $x_1 = \cdots = x_n$.

In this article, we will establish the refinements, additive analogue and some related results of the generalized Ky Fan inequality and arithmetic-geometric mean inequality that are given in paper [1], [5], [6],[7], [8], [10],[11],[12],[14].

3. Ky Fan inequality and related results

Theorem 3.1. *Let $0 < n \leq x_j \leq N \leq \frac{1}{2}$, $j \in \{1, \ldots, n\}$, and $w_j > 0$ with $\sum\limits_{j \in I} w_j = 1$. Then we have the inequalities,*

$$\frac{A_I}{G_I} \geq \left[\frac{A_I}{G_I}\right]^{\dfrac{N^2}{(1-N)^2}} \geq \frac{A'_I}{G'_I} \geq \left[\frac{A_I}{G_I}\right]^{\dfrac{n^2}{(1-n)^2}} \geq 1 \tag{3.1}$$

Proof. The first and the last inequality follows directly from the fact that $\frac{A_I}{G_I} \geq 1, n \in (0, \frac{1}{2}]$ and $N \in (0, \frac{1}{2}]$.

We define the function $g : (0,1) \to \mathbb{R}$, $g(r) = \ln[(\frac{1-r}{r})] + \alpha \ln(r)$ with $\alpha \in R$, we have

$$\begin{aligned} g'(r) &= -\frac{1}{r(1-r)} + \frac{\alpha}{r}, \quad r \in (0,1), \\ g''(r) &= \frac{1}{r^2}\left[\frac{1-2r}{(1-r)^2} - \alpha\right], \quad r \in (0,1). \end{aligned}$$

If we consider the function $h : (0,1) \to \mathbb{R}$, defined as $h(r) = \frac{1-2r}{(1-r)^2}$, then $h'(r) = \frac{2r(r-1)}{(1-r)^4}$, which shows that the function h is monotonically strictly decreasing on $(0,1)$. Consequently for $r \in (n, N)$, we have

$$\frac{1-2N}{(1-N)^2} = h(N) \leq h(r) \leq h(n) = \frac{1-2n}{(1-n)^2}. \tag{3.2}$$

From(3.2) we observe that the function g is strictly convex on (n, N) if $\alpha \leq \frac{1-2N}{(1-N)^2}$. Applying Jensen-Mercer inequality to the function

$$\begin{aligned} &g : (n, N) \to \mathbb{R}, \\ &g(r) = \ln\left[\left(\frac{1-r}{r}\right)\right] + \alpha \ln(r), \end{aligned}$$

with $\alpha \leq \frac{1-2N}{(1-N)^2}$, we conclude that

$$f(c)+f(d)-\sum_{j\in I} w_j f(x_j) \geq f\left(c+d-\sum_{j\in I} w_j x_j\right)$$

$$\ln\left(\frac{1-c}{c}\right)+\alpha\ln c+\ln\left(\frac{1-d}{d}\right)+\alpha\ln d-\sum_{j\in I}\left[\ln\left(\frac{1-x_j}{x_j}\right)+\alpha\ln x_j\right]$$

$$\geq \ln\left(\frac{1-c-d+\sum_{j\in I} w_j x_j}{c+d-\sum_{j\in I} w_j x_j}\right)+\alpha\ln\left(c+d-\sum_{j\in I} w_j x_j\right)$$

$$\ln\frac{G'_I}{G_I}+\alpha\ln G_I \geq \ln\frac{A'_I}{A_I}+\alpha\ln A_I$$

$$\left(\frac{G_I}{A_I}\right)^{\alpha} \geq \frac{A'_I}{A_I}\frac{G_I}{G'_I}$$

$$\left(\frac{G_I}{A_I}\right)^{\alpha-1} \geq \left(\frac{A'_I}{G'_I}\right) \tag{3.3}$$

from (3.3) we observe that this inequality is best possible if we have α is maximal, i.e, $\alpha = \frac{1-2N}{(1-N)2}$, that leads

$$\left(\frac{G_I}{A_I}\right)^{\frac{1-2N}{(1-N)^2}-1} \geq \frac{A'_I}{G'_I}$$

which is equivalent to the second inequality in (3.1). By applying the same technique we established third inequality, by using the function $F(r) = \beta lnr - \ln\left[\frac{(1-r)}{r}\right]$ if $\beta \geq \frac{1-2n}{(1-n)^2}$, then the function is strictly convex on (n, N). □

Remark 3.2. Since the Ky Fan inequality is also equivalent to

$$\frac{A_I}{G_I} \geq \frac{A'_I}{G'_I}$$

then the first part of the inequality may be seen as refinement of the Ky Fan inequality while the second part

$$\frac{A'_I}{G'_I} \geq \left(\frac{A_I}{G_I}\right)^{\frac{n^2}{(1-n)^2}}$$

can be considered as a counter part of the Ky Fan inequality.

Now, by using Lah-Ribarić inequality for convex function which is define in [8]. If $f : [c,d] \subset \mathbb{R} \to \mathbb{R}$ is convex on $[c,d]$, $x_j \in [c,d], w_j \geq 0, j \in \{1,\ldots,n\}$ and

$\sum_{j\in I} w_j = 1$, then

$$f(c)+f(d)-\sum_{j\in I} w_j f(x_j) \leq \frac{d-(c+d-\sum_{j\in I} w_j x_j)}{d-c} f(c)+\frac{(c+d-\sum_{j\in I} w_j x_j - c)}{d-c} f(d) \quad (3.4)$$

Now, we may prove an inequality related to Ky Fan inequality as follow:

Theorem 3.3. *If* $0 < n \leq x_j \leq N \leq \frac{1}{2}$, $j \in \{1,\ldots,n\}$, *and* $w_j > 0$ *and* $\sum_{j\in I} w_j = 1$. *Then we have the inequalities*

$$\left(\frac{1-n}{n^{\frac{n^2}{(1-n)^2}}}\right)^{\frac{N-A_I}{N-n}} \left(\frac{1-N}{N^{\frac{n^2}{(1-n)^2}}}\right)^{\frac{A_I-n}{N-n}} G_I^{\frac{n^2}{(1-n)^2}} \leq G'_I \leq \left(\frac{1-n}{N^{\frac{N^2}{(1-N)^2}}}\right)^{\frac{N-A_I}{N-n}} \left(\frac{1-N}{N^{\frac{N^2}{(1-N)^2}}}\right)^{\frac{A_I-n}{N-n}} G_I^{\frac{N^2}{(1-N)^2}}. \quad (3.5)$$

Proof. From the proof of Theorem (3.1) the function $f : (n,N) \subset (0,\frac{1}{2}] \to \mathbb{R}, f(r) = \ln\left[\frac{1-r}{r}\right] + \frac{(1-2N)}{(1-N)^2} + \ln r$ is strictly convex on (n,N). By applying the Lah-Ribarić inequality for f as above with $c = n$ and $d = N$, we get

$$\ln\left(\frac{1-c}{c}\right)+\frac{1-2N}{(1-N)^2}\ln c+\ln\left(\frac{1-d}{d}\right)+\frac{1-2N}{(1-N)^2}\ln d - \sum_{j\in I} w_j \left[\ln\left(\frac{1-x_j}{x_j}\right)+\frac{1-2N}{(1-N)^2}\ln x_j\right]$$
$$\leq \frac{N-\left(c+d-\sum_{j\in I} w_j x_j\right)}{N-n}\left[\ln\left(\frac{1-n}{n}\right)+\ln(n)^{\frac{1-2N}{(1-N)^2}}\right] + \frac{\left(c+d-\sum_{j\in I} w_j x_j\right)-n}{N-n}\left[\ln\left(\frac{1-N}{N}\right)+\ln(N)^{\frac{1-2N}{(1-N)^2}}\right]$$

which is equivalent to,

$$\ln\left(\frac{G'_I}{G_I}\right)+\frac{1-2N}{(1-N)^2}\ln G_I \leq \frac{N-A_I}{N-n}\left[\ln\left(\frac{1-n}{n}\right)+\ln(n)^{\frac{1-2N}{(1-N)^2}}\right] + \frac{A_I-n}{N-n}\left[\ln\left(\frac{1-N}{N}\right)+\ln(N)^{\frac{1-2N}{(1-N)^2}}\right]$$

which gives us

$$\left(\frac{G'_I}{G_I}\right)[G_I]^{\frac{1-2N}{(1-N)^2}} \leq \left((1-n)n^{\frac{1-2N}{(1-N)^2}-1}\right)^{\frac{N-A_I}{N-n}} \left((1-N)N^{\frac{1-2N}{(1-N)^2}-1}\right)^{\frac{A_I-n}{N-n}}$$

by which we get right-hand inequality in (3.5).

To establish the left-hand inequality, we apply the Lah-Ribarić inequality for the strictly convex function $s : (n,N) \to \mathbb{R}, s(r) = \frac{(1-2n)}{(1-n)^2}\ln r - \ln\left[\frac{(1-r)}{r}\right]$ on (n,N). We omit the details. □

Now, we will prove a result related to the inequality $A'_I \geq G'_I$.

Theorem 3.4. *If $x_j \in (0, \frac{1}{2}], j \in \{1, \ldots, n\}$, then*

$$A'_I \geq G'_I. \tag{3.6}$$

Proof. By using the convex function $\phi(x) = x - \ln(1-x)$ for all $x \in (0, \frac{1}{2}]$ to the Jensen-Mercer inequality, we get

$$\left(c + d - \sum_{j\in I} w_j x_j\right) - \ln\left(1 - c - d + \sum_{j\in I} w_j x_j\right)$$
$$\leq \left(c - \ln(1-c) + d - \ln(1-d) - \sum_{j\in I}(x_j - \ln(1-x_j))\right)$$

$$A_I - \ln(A'_I) \quad \leq \quad A_I - \ln G'_I$$

consequently,

$$A'_I \quad \geq \quad G'_I.$$

□

Now, we will prove Ky Fan inequality via convexity.

Theorem 3.5. *If $x_j \in (0, \frac{1}{2}], j \in \{1, \ldots, n\}$, then*

$$\frac{A'_I}{G'_I} \leq \frac{1}{G_I + G'_I} \leq \frac{A_I}{G_I}. \tag{3.7}$$

Proof. We prove this inequality by using the function $\phi(x) = \dfrac{1}{1+e^x}$, which is strictly convex on $[0, \infty)$ and strictly concave on $(-\infty, 0]$.

To establish the right hand side of the inequality (3.7), we apply convex function to the following inequality,

$$\phi\left(c + d - \sum_{j\in I} w_j y_j\right) \leq \phi(c) + \phi(d) - \sum_{j\in I} w_j \phi(y_j)$$

we define, $y_j = \ln \dfrac{1-x_j}{x_j} \geq 0$ for $j \in \{1, \ldots, n\}, c = \ln\left(\frac{1-c}{c}\right), d = \ln\left(\frac{1-d}{d}\right)$, by which we get,

$$\frac{1}{1 + exp\left(c + d - \sum_{j\in I} w_j y_j\right)} \quad \leq \frac{1}{1+e^c} + \frac{1}{1+e^d} - \sum_{j\in I} \frac{1}{1+e^{y_j}}$$

$$\frac{1}{1+exp\left(\ln\left(\frac{1-c}{c}\right)+\ln\left(\frac{1-d}{d}\right)-\sum_{j\in I}w_j\ln\left(\frac{1-x_j}{x_j}\right)\right)}$$
$$\leq \frac{1}{1+exp\left(\ln\left(\frac{1-c}{c}\right)\right)}+\frac{1}{1+exp\left(\ln\left(\frac{1-d}{d}\right)\right)}$$
$$-\sum_{j\in I}\left(\frac{1}{1+exp\left(\ln\left(\frac{1-x_j}{x_j}\right)\right)}\right)$$

which gives,

$$\frac{1}{1+exp\left(\ln\frac{(1-c)(1-d)}{\left(\prod_{j\in I}(1-x_j)^{w_j}\right)}-\ln\frac{cd}{\prod_{j\in I}x_j^{w_j}}\right)}$$
$$\leq\left(c+d-\sum_{j\in I}w_jx_j\right)$$

consequently, $\dfrac{1}{1+exp\left(\ln\dfrac{G_I'}{G_I}\right)}\leq A_I$

or $\dfrac{1}{G_I+G_I'}\leq\dfrac{A_I}{G_I}$, this gives the right hand side of (3.7).

Now by applying the Jensen-Mercer inequality for the convex function $-f$ on $(-\infty,0]$ with $y_j=\ln\dfrac{x_j}{1-x_j}\leq 0, j\in\{1,\dots,n\}$. We get left side of the inequality (3.7), which completes the proof. □

The following result provides the additive analogue of the Ky Fan inequality.

Theorem 3.6. *If* $x_j\in\left(0,\frac{1}{2}\right], j\in\{1,\dots,n\}$, *then*

$$G_I-G_I'\leq A_I-A_I'.$$

Proof. Without loss of generality we assume that, $0<x_1\leq\cdots\leq x_n=\frac{1}{2}$ with $0<x_1<x_n$.

We have to prove that the the function,

$$\phi\left(x_1,\dots,x_n\right)=\frac{(1-c)\left(1-d\right)}{\prod_{j\in I}\left(1-x_j\right)^{w_j}}-\frac{cd}{\prod\limits_{j\in I}\left(x_j\right)^{w_j}}$$
$$+2\left(c+d-\sum_{j\in I}w_jx_j\right)-1$$

is positive.

Suppose $p \in \{1, \dots, n-1\}$ and $0 < x < x_{p+1}$. We define function

$$\phi_p(x) = \phi(x, \dots, x, x_{p+1}, \dots, x_n)$$

$$\phi_p(x) = \frac{(1-c)(1-d)}{(1-x)^W \prod_{j=p+1}^{n} (1-x_j)^{w_j}} - \frac{cd}{x^W \prod_{j=p+1}^{n} (x_j)^{w_j}} + 2\left(c + d - Wx - \sum_{j=p+1}^{n} w_j x_j\right) - 1$$

on differentiation it yields,

$$\phi_p'(x) = W \frac{(1-c)(1-d)}{(1-x)^{W+1} \prod_{j=p+1}^{n} (1-x_j)^{w_j}} + W\left(\frac{cd}{x^{W+1} \prod_{j=p+1}^{n} (x_j)^{w_j}} - 2\right). \quad (3.8)$$

We deduce from (3.8) that $\phi_p'(x) < 0$. Hence $\phi_p(x)$ is strictly decreasing on $0 < x \le x_{p+1}$, which implies that

$$\phi_p(x_1, \dots, x_n) = \phi_1(x_1) \ge \phi_1(x_2) \quad (3.9)$$

$$\phi_p(x_1, \dots, x_n) = \phi_2(x_2) \ge \phi_2(x_3) \ge \dots \ge \phi_{n-1}(x_{n-1}) \ge \phi_{n-1}(x_n) = 0. \quad (3.10)$$

Since $x_1 < x_n$, we deduce that at least one of the inequalities in (3.9) and (3.10) is strict.

Thus, we get $\phi(x_1, \dots, x_n) > 0$, that completes the proof. □

Now, we will establish some refinements for the Ky Fan inequality which was firstly established by H. Alzer [2] for the unweighted mean, latter J. Rooin [10] established it for weighted mean.

Theorem 3.7. *If $x_j \in (0, \frac{1}{2}], j \in \{1, \dots, n\}$, then*

$$\frac{A_I'}{G_I'} \le \frac{1 - G_I'}{1 - A_I'} \le \frac{A_I}{G_I} \quad (3.11)$$

Proof. The function $\phi(x) = x(1-x)$ is strictly decreasing on $[\frac{1}{2}, \infty)$. Because of $\frac{1}{2} \le G_I' \le A_I' < 1$, we get $\phi(A_I') \le \phi(G_I')$. Which establish the left inequality of (3.11). Since $A_I + A_I' = 1$, and from $A_I' - G_I' \le A_I - G_I$ we obtain that

$$G_I(1 - G_I') \le G_I(2A_I - G_I) \le A_I^2. \quad (3.12)$$

Which gives the right hand side inequality of (3.11). □

Remark 3.8. From the double inequality (3.12), we get the following sharpening of the right-hand side (3.11):

$$\frac{1 - G_I'}{1 - A_I'} \le 2 - \frac{G_I}{A_I} \le \frac{A_I}{G_I}. \quad (3.13)$$

Theorem 3.9. *If $x_j \in \left(0, \frac{1}{2}\right], j \in \{1, \dots, n\}$, then*

$$\frac{A_I'}{G_I'} \le \frac{1 - G_I}{1 - A_I} \le \frac{A_I}{G_I}.$$

Proof. The second side of the inequality valid immediately form

$$0 < G_I \leq A_I \leq \frac{1}{2}$$

and from the fact that $\phi(x) = x(1-x)$ is strictly increasing on $\left(0, \frac{1}{2}\right]$.

To prove first side of the inequality, we may assume,

$$0 < x_1 \leq x_2 \leq \cdots \leq x_n \leq \frac{1}{2},$$

with

$$x_1 < x_n.$$

Then we establish that the function,

$$\phi(x_1, \ldots, x_n) = \left[(1-c) + (1-d) - \frac{1}{W_I}\sum_{j\in I} w_j (1-x_j)\right]^2 - \left(1 - \frac{cd}{\left(\prod_{j\in I} x_j^{w_j}\right)}\right)\left(\frac{(1-c)(1-d)}{\left(\prod_{j\in I}(1-x_j)^{w_j}\right)}\right) \quad (3.14)$$

is negative .

Let $p \in \{1, \ldots, n-1\}$ and $0 < x < x_{p+1}$.

We define a function,

$$\begin{aligned}
\phi_p(x) &= \phi(x, \ldots, x, x_{p+1}, \ldots, x_n) \\
\phi_p(x) &= \left[(1-c) + (1-d) - W(1-x) - \sum_{j=p+1}^{n} w_j (1-x_j)\right]^2 - \\
&\quad \left(1 - \frac{cd}{x^W \prod_{j=p+1}^{n} x_j^{w_j}}\right)\left(\frac{(1-c)(1-d)}{(1-x)^W \prod_{j=p+1}^{n}(1-x_j)^{w_j}}\right)
\end{aligned}$$

is strictly increasing on $x_j \in (0, x_{p+1}]$.

Differentiation of the last result yields,

$$\begin{aligned}
\phi_p'(x) &= W\left[2\left((1-c)(1-d) - W_I(1-x) - \sum_{j=p+1}^{n} w_j (1-x_j)\right)\right] - \\
&\quad W\left(\frac{(1-c)(1-d)}{(1-x)^{W+1}\prod_{j=p+1}^{n}(1-x_j)^{w_j}}\right)\left(1 + \frac{cd}{x^W \prod_{j=p+1}^{n} x_j^{w_j}}\left(\frac{1-2x}{x}\right)\right)
\end{aligned} \quad (3.15)$$

by Arithmetic-Geometric mean inequality, we get $\phi_p'(x) > 0$.

By which we deduce that $\phi_p(x)$ is increasing function for all $x \in (0, x_{p+1}]$. Which implies

$$\phi(x_1, \ldots, x_n) = \phi_1(x_1) \leq \phi_1(x_2) \quad (3.16)$$

$$\phi(x_1, \ldots, x_n) = \phi_2(x_2) \leq \phi_2(x_3) \leq \cdots \leq \phi_{n-1}(x_n) = 0. \quad (3.17)$$

Since ϕ_p is strictly increasing function and $x_1 < x_n$, then we deduce that at least one of the inequality in (3.16) and (3.17) is strict.

Thus we get, $\phi(x_1, \ldots, x_n) < 0$. Which completes the proof. □

Now, we will provide the results related to the inequalities $A_I \geq H_I$ and $A'_I \geq H'_I$.

Theorem 3.10. *If* $x_j \in \left(0, \frac{1}{2}\right], j \in \{1, \ldots, n\}$, *then*

$$(i) A_I \geq H_I.$$

$$(ii) A'_I \geq H'_I.$$

Proof. (i) By using the convex function $\phi(x) = \frac{1}{x}$ for all $x \in (0, \frac{1}{2}]$ in the Jensen-Mercer inequality we get,

$$\begin{aligned}
\frac{1}{c+d-\sum_{j\in I} w_j x_j} &\leq \frac{1}{c} + \frac{1}{d} - \sum_{j\in I} w_j \left(\frac{1}{x_j}\right) \\
A_I &\geq \frac{1}{\frac{1}{c} + \frac{1}{d} - \sum_{j\in I} w_j (\frac{1}{x_j})} \\
A_I &\geq H_I.
\end{aligned}$$

□

Proof. (ii) By using the convex function $\phi(x) = \frac{1}{1-x}$ for all $x \in (0, \frac{1}{2}]$ to Jensen -Mercer inequality we get,

$$\begin{aligned}
\frac{1}{1-(c+d-\sum_{j\in I} w_j x_j)} &\leq \frac{1}{1-c} + \frac{1}{1-d} - \sum_{j\in I} \left(\frac{1}{1-x_j}\right) \\
\frac{1}{(1-c)+(1-d)-\sum_{j\in I} w_j (1-x_j)} &\leq \frac{1}{1-c} + \frac{1}{1-d} - \sum_{j\in I} \left(\frac{1}{1-x_j}\right) \\
A'_I &\geq H'_I.
\end{aligned}$$

□

The arithmetic and harmonic mean inequality was firstly proved by J. Sandor [13]. There are numerous proofs for it. We will establish this inequality by using Jensen-Mercer refinement inequality.

Theorem 3.11. *If* $x_j \in \left(0, \frac{1}{2}\right], j \in \{1, \ldots, n\}$, *then*

$$\frac{1}{A_I} - \frac{1}{A'_I} \leq \frac{1}{H_I} - \frac{1}{H'_I}.$$

Proof. We establish it by applying the Jensen-Mercer inequality to the convex function,

$$\phi(y) = \frac{1}{y} - \frac{1}{1-y}, \quad \left(0 < y \leq \frac{1}{2}\right).$$

By which we obtain,

$$\left(\frac{1}{c+d-\sum_{j\in I} w_j x_j}\right)-\left(\frac{1}{(1-c)+(1-d)-\sum_{j\in I} w_j\left(1-x_j\right)}\right)$$
$$\leq\left(\frac{1}{c}-\frac{1}{1-c}+\frac{1}{d}-\frac{1}{1-d}-\sum_{j\in I} w_j\left(\frac{1}{x_j}-\frac{1}{1-x_j}\right)\right)$$

form the left side of inequality we get,

$$\frac{1}{A_I}-\frac{1}{A_I'}$$

from the right side of the inequality we get,

$$\left[\frac{1}{c}+\frac{1}{d}-\sum_{j\in I} w_j \frac{1}{x_j}\right]-\left[\frac{1}{1-c}+\frac{1}{1-d}-\sum_{j\in I} w_j\left(\frac{1}{1-x_j}\right)\right]$$

$$\frac{1}{\left[\frac{1}{c}+\frac{1}{d}-\sum_{j\in I} w_j \frac{1}{x_j}\right]^{-1}}-\frac{1}{\left[\frac{1}{1-c}+\frac{1}{1-d}-\sum_{j\in I} w_j\left(\frac{1}{1-x_j}\right)\right]^{-1}}$$
$$=\frac{1}{H_I}-\frac{1}{H_I'}$$

that completes the proof. □

Following theorems will provide the result related to the inequality for geometric and harmonic mean.

Theorem 3.12. *If* $x_j \in \left(0, \frac{1}{2}\right], j \in \{1, \ldots, n\}$, *then* $H_I \leq G_I$.

Proof. By using the convex function $\phi(x)=e^x$ for all $x \in[-\infty, \infty)$ for Jensen-Mercer inequality we get

$$exp\left(c+d-\sum_{j\in I} w_j x_j\right) \leq exp\,(c)+exp\,(d)-\sum_{j\in I} w_i exp\left(x_j\right)$$

By replacing c by $\ln\left(\frac{1}{c}\right)$, d by $\ln\left(\frac{1}{d}\right)$ and x_j by $\ln\left(\frac{1}{x_j}\right), j \in\{1, \ldots, n\}$, then we get

$$exp\left(\ln\left(\frac{1}{c}\right)+\ln\left(\frac{1}{d}\right)-\sum_{j\in I} w_j \ln\left(\frac{1}{x_j}\right)\right)$$
$$\leq exp\left(\ln\left(\frac{1}{c}\right)\right)+exp\left(\ln\left(\frac{1}{d}\right)\right)-\sum_{j\in I} w_j exp\left(\ln\left(\frac{1}{x_j}\right)\right)$$

which gives,

$$exp\left(-\ln\left(\frac{cd}{\prod_{j\in I}x_j^{w_j}}\right)\right)\leq\left(\frac{1}{c}+\frac{1}{d}-\sum_{j\in I}w_j\frac{1}{x_j}\right)$$

$$G_I^{-1}\leq\left(\frac{1}{c}+\frac{1}{d}-\sum_{j\in I}w_j\frac{1}{x_j}\right)$$

Hence, we get

$$H_I\leq G_I.$$

□

Theorem 3.13. *If* $x_j\in\left(0,\frac{1}{2}\right]$, $j\in\{1,\ldots,n\}$, *then* $H_I'\leq G_I'$.

Proof. By using the convex function $\phi(x)=e^x$ for all $x\in[0,\frac{1}{2})$ for Jensen-Mercer inequality that is,

$$exp\left(c+d-\sum_{j\in I}w_jx_j\right)\leq exp\,(c)+exp\,(d)-\sum_{j\in I}w_iexp\,(x_j)$$

By using $c=\ln\left(\frac{1}{1-c}\right)$, $d=\ln\left(\frac{1}{1-d}\right)$ and $x_j=\ln\left(\frac{1}{1-x_j}\right)$, $j\in\{1,\ldots,n\}$, then we get

$$exp\left(\ln\left(\frac{1}{1-c}\right)+\ln\left(\frac{1}{1-d}\right)-\sum_{j\in I}w_j\ln\left(\frac{1}{1-x_j}\right)\right)$$
$$\leq exp\left(\ln\left(\frac{1}{1-c}\right)\right)+exp\left(\ln\left(\frac{1}{1-d}\right)\right)-\sum_{j\in I}w_jexp\left(\ln\left(\frac{1}{1-x_j}\right)\right)$$

$$exp\left(-\ln\left(\frac{(1-c)(1-d)}{\prod_{j\in I}(1-x_j)^{w_j}}\right)\right)$$
$$\leq\left(\frac{1}{(1-c)}+\frac{1}{(1-d)}-\sum_{j\in I}w_j\frac{1}{(1-x_j)}\right)$$
$$(G_I')^{-1}\leq\left(\frac{1}{(1-c)}+\frac{1}{(1-d)}-\sum_{j\in I}w_j\frac{1}{(1-x_j)}\right)$$

which gives

$$H_I'\leq G_I'.$$

□

Theorem 3.14. *If* $x_j \in \left(0, \frac{1}{2}\right], j \in \{1, \dots, n\}$, *then*

$$\frac{H'_I}{H_I} \leq \frac{G'_I}{G_I}. \tag{3.18}$$

Proof. Without loss of generality we suppose that $x'_j s$ are not equal and by using the strictly convex function $\phi(y) = \ln\left(\frac{1-y}{y}\right)$, for all $y \in (0, \frac{1}{2}]$.

We set,

$$y = \frac{H_I}{H_I + H'_I}, \quad y \in (0, \frac{1}{2}].$$

Then we get,

$$\begin{aligned}
&\ln\left(\frac{1 - \dfrac{H_I}{H_I + H'_I}}{\dfrac{H_I}{H_I + H'_I}}\right) \\
&= \ln\left(\frac{H'_I}{H_I}\right) \\
&= \ln\left(\frac{\frac{1}{c} + \frac{1}{d} - \displaystyle\sum_{j \in I} w_j \frac{1}{x_j}}{\frac{1}{1-c} + \frac{1}{1-d} - \displaystyle\sum_{j \in I} w_j \frac{1}{1 - x_j}}\right)
\end{aligned}$$

$$\begin{aligned}
= \ln\left(\frac{1}{c} + \frac{1}{d} - \sum_{j \in I} w_j \frac{1}{x_j}\right) - \ln\left(\frac{1}{1-c} + \frac{1}{1-d} - \sum_{j \in I} w_j \frac{1}{1-x_j}\right)& \\
\leq \ln\left(\frac{1}{cd}\right) - \ln\left(\sum_{j=1}^{n} w_j \frac{1}{x_j}\right)& \\
- \left(\ln \frac{1}{(1-c)(1-d)} - \ln\left(\sum_{j \in I} w_j \frac{1}{1-x_j}\right)\right)&
\end{aligned}$$

from the right side of the inequality we get,

$$\begin{aligned}
- \left(\ln(cd) - \ln\left(\prod_{j \in I} (x_j)^{w_j}\right)\right)& \\
+ \ln((1-c)(1-d)) - \ln\left(\prod_{j \in I} (1 - x_j)^{w_j}\right)&
\end{aligned}$$

which is equivalent to

$$\ln\left(\frac{G'_I}{G_I}\right).$$

Taking exponential on both sides we get inequality (3.18), that completes the proof. □

Theorem 3.15. *If $x_j \in (0, \frac{1}{2}]$. then,*

$$\frac{H'_I}{G'_I} \leq \frac{1}{G_I + G'_I} \leq \frac{H_I}{G_I}. \tag{3.19}$$

Proof. To prove the left hand inequality we apply convex function $\phi(x) = \dfrac{1}{e^x} + 1$ on $(-\infty, \infty]$ to the Jensen-Mercer inequality that is,

$$\frac{1}{exp\left(c + d - \sum_{j\in I} w_j x_j\right)} + 1 \leq \left(\frac{1}{exp(c)} + 1\right) + \left(\frac{1}{exp(d)} + 1\right) - \sum_{j\in I} w_j \left(\frac{1}{exp(x_j)} + 1\right)$$

by using $x_j = \ln(\frac{1-x_j}{x_j}) \geq 0, j \in \{1, \dots, n\}, c = \ln(\frac{1-c}{c})$ and $d = \ln(\frac{1-d}{d})$

$$\frac{1}{exp\left(\ln\left(\frac{1-c}{c}\right) + \ln\left(\frac{1-d}{d}\right) - \sum_{j\in I} w_j \ln\left(\frac{1-x_j}{x_j}\right)\right)} + 1$$
$$\leq \frac{1}{exp\left(\ln(\frac{1-c}{c})\right)} + 1 + \frac{1}{exp\left(\ln\left(\frac{1-d}{d}\right)\right)} + 1 - \sum_{j\in I}\left[\frac{1}{exp\left(\left(\ln \frac{1-x_j}{x_j}\right)\right)} + 1\right]$$

$$\frac{1}{exp\left[\ln\left(\frac{G'_I}{G_I}\right)\right]} + 1 \leq \left(\frac{1}{1-c} + \frac{1}{1-d} - \sum_{j\in I} w_j \frac{1}{1-x_j}\right)$$
$$\frac{G_I + G'_I}{G'_I} \leq \left(\frac{1}{1-c} + \frac{1}{1-d} - \sum_{j\in I} w_j \frac{1}{1-x_j}\right)$$
$$\frac{H'_I}{G'_I} \leq \frac{1}{G_I + G'_I}.$$

To prove right-hand of the inequality (3.19) we apply Jensen-Mercer inequality to the convex function $-f$ on $(-\infty, \infty]$ with $x_j = \ln\left(\frac{x_j}{1-x_j}\right) \leq 0, j \in \{1, \dots, n\}$, $c = \ln\left(\frac{c}{1-c}\right)$, $d = \ln\left(\frac{d}{1-d}\right)$. □

Finally, we will improve bounds of the inequality

$$0 < \frac{A'_I - G'_I}{A_I - G_I} < 1$$

and we will show that the inequalities

$$\min_{1\le j\le n} \frac{x_j}{1-x_j} < \frac{G'_I - A'_I}{G_I - A_I} < \max_{1\le j\le n} \frac{x_j}{1-x_j}$$

is authentic for all $x_j \in (0, \frac{1}{2}]$ with $j \in \{1, \dots, n\}; n \ge 2$. In order to establish above inequality we required two inequalities containing the geometric mean G_I and G'_I. For this we will need the following lemmas.

Lemma 3.16. *Let $x_j \in (0, \frac{1}{2}]$ with $j \in \{1, \dots, n\}; n \ge 2$ and let $M = \max\limits_{1\le j\le n}\{x_j\}$ and $m = \min\limits_{1\le j\le n}\{x_j\}$. Then we have*

$$1 \le \frac{M}{m} G_I + \frac{1-M}{1-m} G'_I.$$

Proof. Without loss of generality, we may assume that

$$m = x_1 \le x_2 \le \cdots \le x_{n-1} \le x_n = M, \quad m < M.$$

We have to prove that

$$\phi(x_1, \dots, x_n) = \frac{1-x_n}{1-x_1} \frac{(1-c)(1-d)}{\left(\prod_{j\in I}(1-x_j)^{w_j}\right)} + \frac{x_n}{x_1} \frac{ab}{\left(\prod_{j\in I} x_j^{w_j}\right)} > 1$$

Let $p \in \{1, \dots, n-1\}$ and $0 < x < x_{p+1}$, we define a function

$$\phi_p(x) = \phi(x, \dots, x, x_{p+1}, \dots, x_n) \tag{3.20}$$

$$\phi_p(x) = \frac{(1-x_n)}{(1-x)^{W+1}} \frac{(1-c)(1-d)}{\prod\limits_{j=p+1}^{n}(1-x_j)^{w_j}} + \frac{x_n}{x^{W+1}} \frac{cd}{\prod\limits_{j=p+1}^{n} x_j^{w_j}}$$

differentiation yields,

$$\phi'_p(x) = (W+1)\left[\frac{(1-x_n)}{(1-x)^{W+2}} \frac{(1-c)(1-d)}{\prod\limits_{j=p+1}^{n}(1-x_j)^{w_j}} - \frac{x_n}{x^{W+2}} \frac{cd}{\prod\limits_{j=p+1}^{n} x_j^{w_j}}\right]$$

$$\phi_p'(x) = (W+1)\frac{(1-x_n)}{x^{W+2}\prod_{1+p}^{n} x^{W+2}}$$
$$\left[\left(\frac{x}{1-x}\right)^{W+2}\prod_{j=p+1}^{n}\left(\frac{x_j}{1-x_j}\right)^{w_j} - \frac{x_n}{1-x_n}\frac{cd}{(1-c)(1-d)}\right] \tag{3.21}$$

Using $\frac{x}{1-x} < \frac{x_n}{1-x_n}$ and $\left(\frac{x}{1-x}\right)^{W+1} < \left(\frac{x_n}{1-x_n}\right)$, we deduce that $\phi_p'(x) < 0$. Hence, $\phi_p(x)$ strictly decreasing on$(0, \frac{1}{2}]$, which implies

$$\phi(x_1, \dots, x_n) = \phi_1(x_1) \geq \phi_2(x_2) \tag{3.22}$$
$$\phi(x_1, \dots, x_n) = \phi_2(x_2) \geq \phi_2(x_3) \geq \dots \geq \phi_{n-1}(x_{n-1}) \geq \phi_{n-1}x_n = 1. \tag{3.23}$$

Since $x_1 < x_n$. We deduce that at least one of the inequalities is strict in (3.22) and (3.23). Thus, we get $\phi(x_1, \dots, n) > 1$. □

Lemma 3.17. *Let $x_j \in (0, \frac{1}{2}]$ with $j \in \{1, \dots, n\}; n \geq 2$ be real numbers and let $M = \max\limits_{1\leq j\leq n}\{x_i\}$ and $\min\limits_{1\leq j\leq n}\{x_i\}$. Then we have,*

$$1 \leq \frac{m}{M}G_I + \frac{1-m}{1-M}G_I'.$$

Proof. Let $m = x_1 \leq x_2 \leq \dots \leq x_{n-1} \leq x_n = M, \quad m < M$.

$$\phi(x_1, \dots, x_n) = \frac{1-x_1}{1-x_n}\frac{(1-c)(1-d)}{\prod\limits_{j\in I}(1-x_j)^{w_j}} + \frac{x_1}{x_n}\frac{cd}{\prod\limits_{j\in I}x_j^{w_j}}$$

Let $p \in \{1, \dots, n-1\}$ and $0 < x < x_{p+1}$. We define a function

$$\phi_p(x) = \phi(x, \dots, x, x_{p+1}, \dots, x_n)$$

$$\phi_p(x) = \frac{(1-x)}{(1-x_n)}\frac{(1-c)(1-d)}{(1-x)^W\prod\limits_{j=p+1}^{n}(1-x_j)^{w_j}} + \frac{x}{x_n}\frac{cd}{x^W\prod\limits_{j=p+1}^{n}x_j^{w_j}}$$

$$\phi_p(x) = \left(\frac{(1-x)^{1-W}}{(1-x_n)}\right)\left(\frac{(1-c)(1-d)}{\prod_{j=p+1}^{n}(1-x_j)^{w_j}}\right) + \frac{x^{1-W}}{x_n}\frac{cd}{\prod_{j=p+1}^{n}x_j^{w_j}}$$

differentiation of the last result yields,

$$\phi_p'(x) = (1-W)\left[\frac{-(1-x)^{-W}}{(1-x_n)}\frac{(1-c)(1-d)}{\prod_{j=p+1}^{n}(1-x_j)^{w_j}} + \frac{x^{-W}}{x_n}\frac{ab}{\prod_{j=p+1}^{n}x_j^{w_j}}\right]$$

$$\phi_p'(x) = (1-W)\frac{(1-c)(1-d)}{x_n x^W}\frac{1}{\prod_{j=p+1}^n x_j^{w_j}}$$
$$\left[\frac{cd}{(1-c)(1-d)} - \left(\frac{x}{1-x}\right)^W \frac{x_n}{1-x_n}\left(\prod_{j=p+1}^n \frac{x_j}{1-x_j}\right)^{w_j}\right]$$

If we let $c = x$ and $x_n = d$ then,

$$\phi_p'(x) = \frac{(1-W)(1-x)^{1-W}}{\left(\prod_{j=p+1}^n x_j^{w_j}\right)}\left[\left(\frac{x}{1-x}\right)^{1-W} - \left(\prod_{j=p+1}^n \left(\frac{x_j}{1-x_j}\right)^{w_j}\right)\right]$$

since

$$\left(\frac{x}{1-x}\right)^{1-W} < \prod_{j=p+1}^n \left(\frac{x_j}{1-x_j}\right)^{w_j}$$

therefore we conclude $\phi_p'(x) < 0$. Hence $\phi_p(x)$ is strictly decreasing function on $0 < x \le x_{p+1}$, which implies

$$\phi(x_1,\dots,x_n) = \phi_1(x_1) \ge \phi_1(x_2) \tag{3.24}$$
$$\phi(x_1,\dots,x_n) = \phi_2(x_2) \ge \phi_2(x_3) \ge \cdots \ge \phi_{n-1}(x_{n-1}) \ge \phi_{n-1}(x_n) = 1. \tag{3.25}$$

Since $x_1 < x_n$ from which we deduce that at least one of inequality in above inequalities (3.24) and (3.25) is strict. Thus we get $\phi(x_1,\dots,x_n) > 1$. □

Theorem 3.18. *Let $x_j \in (0, \frac{1}{2}], j \in \{1,\dots,n\}; n \ge 2$ be real numbers, and let $M = \max\limits_{1\le j\le n}\{x_i\}$ and $m = \min\limits_{1\le j\le n}\{x_j\}$. Then we have,*

$$\frac{m}{1-m} < \frac{G_I' - A_I'}{G_I - A_I} < \frac{M}{1-M}. \tag{3.26}$$

Proof. To prove the right-hand side of the inequality (3.26) we suppose

$$m = x_1 \le x_2 \le \cdots \le x_{n-1} \le x_n = M, \quad m < M.$$

Then we have to prove that the function

$$\phi(x_1,\dots,x_n) = (1-x_n)\left[\frac{(1-c)(1-d)}{\left(\prod_{j\in I}(1-x_j)^{w_j}\right)}\left(c+d-\sum_{j\in I} w_j x_j\right) - 1\right] +$$
$$x_n\left[\left(c+d-\sum_{j\in I} w_j x_j\right) - \frac{cd}{\left(\prod_{j\in I} x_j^{w_j}\right)}\right] < 0$$

Let $p \in \{1,\dots,n-1\}$ and $0 < x < x_{p+1}$. we define a function

$$\phi(x) = \phi(x,\dots,x,x_{p+1},\dots,x_n)$$

$$\phi_p(x) = (1-x_n)\left[\frac{(1-c)(1-d)}{(1-x)^W \prod_{j=1+p}^{n}(1-x_j)^{w_j}} + \left(c+d-Wx-\sum_{j=p+1}^{n} w_j x_j\right) - 1\right] + x_n\left[\left(c+d-Wx-\sum_{j=p+1}^{n} w_j x_j\right) - \frac{cd}{x^W \prod_{j=1+p}^{n} x_j^{w_j}}\right] > 0$$

differentiation of the last result yields,

$$\phi_p'(x) = W\left[\frac{(1-x_n)(1-c)(1-d)}{(1-x)^{1+W}\prod\limits_{j=1+p}^{n}(1-x_j)^{w_j}} + \frac{x_n cd}{x^{1+W}\prod\limits_{j=1+p}^{n} x_j^{w_j}} - 1\right],$$

by using Lemma (3.16) we get $\phi_p'(x) > 0$ for all $x \in (0, x_{p+1})$. This implies

$$\phi(x_1,\dots,x_n) = \phi_1(x_1) \le \phi_1(x_2) \tag{3.27}$$

$$\phi(x_1,\dots,x_n) = \phi_2(x_2) \le \phi_2(x_3) \ge \cdots \ge \phi_{n-1}(x_{n-1}) \ge \phi_{n-1}(x_n) = 0. \tag{3.28}$$

Since ϕ_p is strictly increasing, we deduce from $x_1 < x_n$ that at least one of the inequalities in (3.27) and (3.28) is strict. Thus we get $\phi(x_1,\dots,x_n) < 0$.

To establish left-hand side of the inequality (3.26) we assume

$$m = x_n \le x_{n-1} \le \cdots \le x_2 \le x_1 = M, \quad m < M.$$

We have to establish that

$$\phi(x_1,\dots,x_n) = (1-x_n)\left[1-\left(c+d-\sum_{j\in I} w_j x_j\right)\right] - \frac{(1-c)(1-d)}{\prod_{j\in I}(1-x_j)^{w_j}} - x_n\left[\left(c+d-\sum_{j\in I} w_j x_j\right) - \frac{cd}{\prod_{j\in I} x_j^{w_j}}\right]$$

is positive. Let $1 \le p \le n-1$, $x_{p+1} < x \le \frac{1}{2}$. We define a function

$$\phi_p(x) \quad = \quad J(x,\dots,x,x_{p+1},\dots,x_n)$$

$$\phi_p(x) = (1-x_n)\left[1-\left(c+d-Wx-\frac{1}{W_I}\sum_{j=1+p}^{n} w_j x_j\right) - \left(\frac{(1-c)(1-d)}{(1-x)^W\left(\prod_{j=p+1}^{n}(1-x_j)^{w_j}\right)}\right)\right] - x_n\left[\left(c+d-Wx-\sum_{j=1+p}^{n} w_j x_j\right) - \left(\frac{cd}{x^W\left(\prod_{j=1+p}^{n} x_j^{w_j}\right)}\right)\right]$$

and

$$\phi_p'(x) = (1-x_n)\left[W - \frac{W(1-c)(1-d)}{(1-x)^{1+W}\prod_{j=p+1}^{n}(1-x_j)^{w_j}}\right] - x_n\left[-W + \frac{Wcd}{x^{1+W}\prod_{j=1+p}^{n} x_j^{w_j}}\right]$$

$$\phi_p'(x) = W\left[1 - \frac{(1-x_n)(1-c)(1-d)}{(1-x)^{1+W}\prod\limits_{j=p+1}^{n}(1-x_j)^{w_j}} - \frac{x_n cd}{x^{1+W}\prod\limits_{j=1+p}^{n} x_j^{w_j}}\right]$$

By using Lemma (3.17) we get $\phi_p'(x) < 0$, which implies

$$\begin{aligned}\phi(x_1,\ldots,x_n) &= \phi_1(x_1) \le \phi_2(x_2)\\ \phi(x_1,\ldots,x_n) &= \phi_2(x_2) \le \phi_2(x_3) \le \cdots \le \phi_{n-1}x_{n-1} \le \phi_{n-1}x_n = 0.\end{aligned}$$

Hence $\phi_p(x)$ is strictly decreasing, we obtain $\phi(x_1,\ldots,x_n) < 0$. That completes the proof. □

4. Conclusions

Precise proofs related to Ky Fan inequality were presented, that provided refinements and extensions of Ky Fan inequality. Moreover, some variants and related results of generalized Ky Fan inequality including the improvement of the bounds of the inequality

$$0 < \frac{A_I' - G_I'}{A_I - G_I} < 1$$

are provided and showed that the inequalities

$$\min_{1\le j\le n} \frac{x_j}{1-x_j} < \frac{G_I' - A_I'}{G_I - A_I} < \max_{1\le j\le n} \frac{x_j}{1-x_j}$$

is authentic for all $x_j \in (0, \frac{1}{2}], j \in \{1,\ldots,n\}; n \ge 2$.

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