FUNCTIONAL INEQUALITIES FOR INCOMPLETE BETA AND GAMMA FUNCTIONS

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Abstract. In the present paper several new functional inequalities are proved.

1. Introduction

The Gamma and Beta functions are respectively defined by
\[ \Gamma(a) = \int_0^\infty t^{a-1}e^{-t} \, dt, \quad a > 0, \quad (1.1) \]
\[ B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} \, dt = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}} \, dt, \quad a, b > 0. \quad (1.2) \]

For \(0 \leq x \leq \infty\), the incomplete Gamma and Beta functions are respectively defined by
\[ \Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t} \, dt, \quad a > 0, \quad (1.3) \]
\[ B(a, b, x) = \int_x^\infty \frac{t^{a-1}}{(1+t)^{a+b}} \, dt, \quad a, b > 0. \quad (1.4) \]

In [1], Ismail and Laforgia proved the following results

**Theorem 1.1.** For a fixed \(a > 0\), the function
\[ f(x) := \frac{\Gamma(a, x)}{\Gamma(a)} \quad (1.5) \]
satisfies the inequality
\[ f(x) f(y) \geq f(x + y), \quad x, y \geq 0, \quad (1.6) \]
when \(a \geq 1\). If \(a \leq 1\), the inequality (1.6) is reversed.
Theorem 1.2. Let
\[ h(x) := \int_{x}^{\infty} u(t) e^{-t} dt, \quad f(x) := h(x)/h(0). \] (1.7)
If \( u(x+y)/u(x) \) is non-increasing in \( x \) on \((0, \infty)\) for every \( y > 0 \), then \( f \) satisfies (1.6). If \( u(x+y)/u(x) \) is non-decreasing in \( x \) for every \( y > 0 \), then (1.6) is reversed.

In fact Theorem 1.2 is a generalization of Theorem 1.1.

The aim of this paper is to present three general theorems concerning inequalities of the type
\[ f(x)f(y) \geq f(x+y), \quad f(x)f(y) \geq f(xy), \quad f(xy) \geq f(x+y-1), \]
and hence as a consequence to obtain some new results by applying these theorems to some special functions.

2. Results

Theorem 2.1. Let \( g(x) \geq 0, \ 1 \leq x \leq \infty. \) Define
\[ f(x) = \int_{x}^{\infty} g(t) dt, \quad F(x) = f(x)/f(1). \] (2.1)
If \( g(xy)/g(x) \) is non-increasing in \( x \) on \((0, \infty)\) for every \( y > 0 \), then the following inequality holds
\[ F(x)F(y) \geq F(xy). \] (2.2)
If \( g(xy)/g(x) \) is non-decreasing in \( x \) on \((0, \infty)\) for every \( y > 0 \), then (2.2) is reversed.

Proof. Set
\[ G(x) = F(x)F(y) - F(xy). \]
On keeping \( y \) fixed, we have
\[ G'(x) = F'(x)F(y) - yF'(xy) \]
\[ = \frac{g(x)f(y)}{f(1)} \left( \frac{y}{F(y)} \frac{g(xy)}{g(x)} - 1 \right). \]
Set
\[ H(x) = \frac{y}{F(y)} \frac{g(xy)}{g(x)} - 1. \]
Since
\[ G(1) = F(1)F(y) - F(y) = 0, \quad \lim_{x \to \infty} G(x) = 0, \] (2.3)
then by Roll’s theorem there exists a point \( p \in (0, \infty) \) such that \( G'(p) = 0 \). Now, \( H(x) \) is decreasing in \( x \) on \((0, p)\) with \( H(p) = 0 \), then \( H(x) > 0 \) on \((0, p)\). Therefore on \((0, p)\), \( G'(x) > 0 \), being a positive multiple of \( H(x) \). This fact with (2.3) shows that \( G(x) \) is increasing on \((0, p)\) vanishing at \( p \) and decreasing on \((p, \infty)\). Therefore \( G(x) \geq 0 \).

The following is a simple generalization of Theorem 1.2.
**Theorem 2.2.** Let \( g(x) \geq 0, 0 \leq x \leq \infty \). Define
\[
\begin{align*}
    f(x) &= \int_x^\infty g(t) \, dt, \\
    F(x) &= f(x) / f(0).
\end{align*}
\]  
(2.4)

If \( g(x+y)/g(x) \) is non-increasing in \( x \) on \((0, \infty)\) for every \( y > 0 \), then the following inequality holds
\[
    F(x) F(y) \geq F(x+y).
\]  
(2.5)

If \( g(x+y)/g(x) \) is non-decreasing in \( x \) on \((0, \infty)\) for every \( y > 0 \), then (2.5) is reverses.

**Proof.** Set
\[
    G(x) = F(x)F(y) - F(x+y).
\]

On keeping \( y \) fixed, we have
\[
    G'(x) = F'(x)F(y) - F'(x+y)
\]
\[
= g(x)F(y)
\]
\[
= \frac{F(y)}{f(0)} \left( \frac{1}{g(x)} g(x+y) - 1 \right).
\]

Set
\[
    H(x) = \frac{1}{F(y)} \frac{g(x+y)}{g(x)} - 1.
\]

Since
\[
    G(0) = F(0)F(y) - F(y) = 0, \quad \lim_{x \to \infty} G(x) = 0,
\]  
(2.6)

then by Roll’s theorem there exists a point \( p \in (0, \infty) \) such that \( G'(p) = 0 \). Now, \( H(x) \) is decreasing in \( x \) on \((0, p)\) with \( H(p) = 0 \), then \( H(x) > 0 \) on \((0, p)\). Therefore on \((0, p)\), \( G'(x) > 0 \), being a positive multiple of \( H(x) \). This fact with (2.6) shows that \( G(x) \) is increasing on \((0, p)\) vanishing at \( p \) and decreasing on \((p, \infty)\). Therefore \( G(x) \geq 0 \). 

**Theorem 2.3.** Let \( g(x) \geq 0, 1 \leq x \leq \infty \). Define
\[
\begin{align*}
    f(x) &= \int_x^\infty g(t) \, dt, \\
    F(x) &= f(x) / f(1).
\end{align*}
\]  
(2.7)

If \( g(x+y-1)/g(x) \) is non-increasing in \( x \) on \((0, \infty)\) for every \( y > 0 \), then the following inequality holds
\[
    F(xy) \geq F(x+y-1).
\]  
(2.8)

If \( g(x+y-1)/g(x) \) is non-decreasing in \( x \) on \((0, \infty)\) for every \( y > 0 \), then (2.8) is reverses.

**Proof.** Set
\[
    G(x) = F(xy) - F(x+y-1).
\]
On keeping \( y \) fixed, we have
\[
G'(x) = y F'(xy) - y F'(x + y - 1)
\]
\[
= y \frac{g(xy)}{f(1)} \left( \frac{g(x + y - 1)}{y g(xy)} - 1 \right).
\]
Set
\[
H(x) = \frac{g(x + y - 1)}{y g(xy)} - 1.
\]
Since
\[
G(1) = F(y) - F(y) = 0, \quad \lim_{x \to \infty} G(x) = 0, \quad (2.9)
\]
then by Roll’s theorem there exists a point \( p \in (0, \infty) \) such that
\[
G'(p) = 0.
\]
Now,
\[
H(x) \text{ is decreasing in } x \text{ on } (0, p) \text{ with } H(p) = 0,
\]
then
\[
H(x) > 0 \text{ on } (0, p).
\]
Therefore
\[
G'(x) > 0 \text{ on } (0, p), \text{ being a positive multiple of } H(x).
\]
This fact with \((2.9)\) shows that \( G(x) \) is increasing on \((0, p)\) vanishing at \( p \) and decreasing on \((p, \infty)\). Therefore \( G(x) \geq 0 \).

3. Applications

**Remark.** Theorem 1.1 follows from Theorem 2.2 by putting
\[
g(t) = t^{a-1} e^{-t}, \quad a > 0.
\]

**Theorem 3.1.** Let \( a, b > 0, a < 1 \). Then the function \( F(x) = B(a, b, x)/B(a, b) \) satisfy the inequality
\[
F(x)F(y) \leq F(x + y), \quad x, y > 0.
\]
**Proof.** The proof follows from Theorem 2.2 by putting
\[
f(x) = \int_{x}^{\infty} \frac{t^{a-1} e^{-t}}{(1 + t)^{a+b}} dt.
\]
In fact,
\[
\frac{g(x + y)}{g(x)} = \left( 1 + \frac{y}{x} \right)^{a-1} \left( 1 + \frac{y}{1 + x} \right)^{-a-b},
\]
is non-decreasing, being the product of two non-decreasing functions. \( \Box \)

**Theorem 3.2.** Let \( a > 0, x \geq 0, y > 1 \). Then the function \( F(x) = \Gamma(a, x)/\Gamma(a) \) satisfy the inequality \((2.2)\). If \( y \in (0, 1) \), then \((2.2)\) reverses.

**Proof.** The proof follows from theorem 2.1, by putting
\[
f(x) = \int_{x}^{\infty} t^{a-1} e^{-t} dt.
\]
We have
\[
\frac{g(xy)}{g(x)} = y^{a-1} e^{-x(y-1)},
\]
which is non-decreasing for \( y > 1 \), and non-increasing for \( 0 < y < 1 \). \( \Box \)
Theorem 3.3. Let $x \geq 0, a, b > 0, y > 1$. Then the function $F(x) = B(a, b, x) / B(a, b)$ satisfy the inequality \ref{inequality}. If $y \in (0, 1)$, then \ref{inequality} reverses.

Proof. The proof follows from theorem 2.1, by putting

$$f(x) = \int_x^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt, \quad x \geq 0.$$ 

In fact

$$\frac{f'(xy)}{f'(x)} = y^{a-1} \left( \frac{1+x}{1+xy} \right)^{a+b},$$

which is non-increasing in $x$ as

$$\left( \frac{1+x}{1+xy} \right)' = \frac{1-y}{(1+xy)^2} \leq 0, \quad \text{for } y > 1.$$

\[\blacksquare\]

Theorem 3.4. Let $x \geq 0, a > 0$. Then the function $F(x) = \Gamma(a, x) / \Gamma(a)$ satisfy the inequality \ref{inequality} provided $a, y \in (1, \infty)$ or $a, y \in (0, 1)$. If $a \in (0, 1), y \in (1, \infty)$ or $a \in (1, \infty), y \in (0, 1)$ then \ref{inequality} reverses.

Proof. The proof follows from theorem 2.3, by putting

$$F(x) = \int_x^\infty t^{a-1} e^{-t} dt, \quad x \geq 0.$$ 

We have

$$\frac{g(x+y-1)}{g(x)} = e^{1-y} \left( 1 + \frac{y-1}{x} \right)^{-a},$$

which implies that $\frac{g(x+y-1)}{g(x)}$ is non-increasing whenever

$$a, y \in (1, \infty) \text{ or } a, y \in (0, 1)$$

and it is non-decreasing whenever

$$a \in (0, 1), y \in (1, \infty) \text{ or } a \in (1, \infty), y \in (0, 1).$$

\[\blacksquare\]

Theorem 3.5. Let $b > 0, x \geq 0$. Then the function $F(x) = B(a, b, x) / B(a, b)$ satisfy the inequality \ref{inequality}. If $0 < y < 1, a > 1$. If $y > 1, 0 < a < 1$, the inequality \ref{inequality} reverses.

Proof. The proof follows from theorem 2.3 by putting

$$f(x) = \int_x^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt, \quad x \geq 0.$$ 

We have

$$\frac{g(x+y-1)}{g(xy)} = y^{1-a} \left( 1 + \frac{y-1}{x} \right)^{-a} \left( \frac{1+xy}{x+y} \right)^{a+b},$$

$$\left( \frac{1+xy}{x+y} \right)' = \frac{y^2-1}{(x+y)^2}.$$
which shows that $\frac{g(x + y - 1)}{g(xy)}$ is non-increasing for $0 < y < 1$, $a > 1$ and it is non-decreasing for $y > 1$, $a < 1$.

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REFERENCES


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