Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://www.ilirias.com
Volume 1 Issue 1(2010), Pages 10-15.

FUNCTIONAL INEQUALITIES FOR INCOMPLETE BETA AND GAMMA FUNCTIONS

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Abstract. In the present paper several new functional inequalities are proved.

1. Introduction

The Gamma and Beta functions are respectively defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt, \qquad a > 0, \tag{1.1}$$

$$B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt, \qquad a, b > 0. \tag{1.2}$$

For $0 \leq x \leq \infty$, the incomplete Gamma and Beta functions are respectively defined by

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt, \qquad a > 0, \tag{1.3}$$

$$B(a,b,x) = \int_x^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt, \qquad a, b > 0. \tag{1.4}$$

In [1], Ismail and Laforgia proved the following results

Theorem 1.1. *For a fixed $a > 0$, the function*

$$f(x) := \Gamma(a,x)/\Gamma(a) \tag{1.5}$$

satisfies the inequality

$$f(x) f(y) \geq f(x+y), \qquad x, y \geq 0, \tag{1.6}$$

when $a \geq 1$. If $a \leq 1$, the inequality (1.6) is reversed.

2000 *Mathematics Subject Classification.* 33B15, 33C05.
Key words and phrases. Incomplete function; integral inequality.

Submitted August 8, 2010. Published October 16, 2010.

Theorem 1.2. *Let*

$$h(x) := \int_x^\infty u(t)\, e^{-t} dt, \qquad f(x) := h(x)/h(0). \tag{1.7}$$

If $u(x+y)/u(x)$ is non-increasing in x on $(0,\infty)$ for every $y>0$, then f satisfies (1.6). If $u(x+y)/u(x)$ is non-decreasing in x for every $y>0$, then (1.6) is reversed.

In fact Theorem 1.2 is generalization of Theorem 1.1.

The aim of this paper is to present three general theorems concerning inequalities of the type $f(x)f(y) \geq f(x+y)$, $f(x)f(y) \geq f(xy)$, $f(xy) \geq f(x+y-1)$, and hence as a consequence to obtain some new results by applying these theorems to some special functions.

2. Results

Theorem 2.1. *Let $g(x) \geq 0$, $1 \leq x \leq \infty$. Define*

$$f(x) = \int_x^\infty g(t)\, dt, \qquad F(x) = f(x)/f(1). \tag{2.1}$$

If $g(xy)/g(x)$ is non-increasing in x on $(0,\infty)$ for every $y>0$, then the following inequality holds

$$F(x)\, F(y) \geq F(xy). \tag{2.2}$$

If $g(xy)/g(x)$ is non-decreasing in x on $(0,\infty)$ for every $y>0$, then (2.2) is reverses.

Proof. Set

$$G(x) = F(x)F(y) - F(xy).$$

On keeping y fixed, we have

$$\begin{aligned} G'(x) &= F'(x)F(y) - yF'(xy) \\ &= \frac{g(x)F(y)}{f(1)} \left(\frac{y}{F(y)} \frac{g(xy)}{g(x)} - 1 \right). \end{aligned}$$

Set

$$H(x) = \frac{y}{F(y)} \frac{g(xy)}{g(x)} - 1.$$

Since

$$G(1) = F(1)F(y) - F(y) = 0, \quad \lim_{x\to\infty} G(x) = 0, \tag{2.3}$$

then by Roll's theorem there exists a point $p \in (0,\infty)$ such that $G'(p) = 0$. Now, $H(x)$ is decreasing in x on $(0,p)$ with $H(p) = 0$, then $H(x) > 0$ on $(0,p)$. Therefore on $(0,p)$, $G'(x) > 0$, being a positive multiple of $H(x)$. This fact with (2.3) shows that $G(x)$ is increasing on $(0,p)$ vanishing at p and decreasing on (p,∞). Therefore $G(x) \geq 0$. □

The following is a simple generalization of Theorem 1.2.

Theorem 2.2. *Let $g(x) \geq 0$, $0 \leq x \leq \infty$. Define*

$$f(x) = \int_x^\infty g(t)\,dt, \qquad F(x) = f(x)/f(0). \tag{2.4}$$

If $g(x+y)/g(x)$ is non-increasing in x on $(0,\infty)$ for every $y > 0$, then the following inequality holds

$$F(x)F(y) \geq F(x+y). \tag{2.5}$$

If $g(x+y)/g(x)$ is non-decreasing in x on $(0,\infty)$ for every $y > 0$, then (2.5) is reverses.

Proof. Set

$$G(x) = F(x)F(y) - F(x+y).$$

On keeping y fixed, we have

$$\begin{aligned} G'(x) &= F'(x)F(y) - F'(x+y) \\ &= \frac{g(x)F(y)}{f(0)}\left(\frac{1}{F(y)}\frac{g(x+y)}{g(x)} - 1\right). \end{aligned}$$

Set

$$H(x) = \frac{1}{F(y)}\frac{g(x+y)}{g(x)} - 1.$$

Since

$$G(0) = F(0)F(y) - F(y) = 0, \quad \lim_{x\to\infty} G(x) = 0, \tag{2.6}$$

then by Roll's theorem there exists a point $p \in (0,\infty)$ such that $G'(p) = 0$.Now, $H(x)$ is decreasing in x on $(0,p)$ with $H(p) = 0$, then $H(x) > 0$ on $(0,p)$. Therefore on $(0,p)$, $G'(x) > 0$, being a positive multiple of $H(x)$. This fact with (2.6) shows that $G(x)$ is increasing on $(0,p)$ vanishing at p and decreasing on (p,∞). Therefore $G(x) \geq 0$. □

Theorem 2.3. *Let $g(x) \geq 0$, $1 \leq x \leq \infty$. Define*

$$f(x) = \int_x^\infty g(t)\,dt, \qquad F(x) = f(x)/f(1). \tag{2.7}$$

If $g(x+y-1)/g(x)$ is non-increasing in x on $(0,\infty)$ for every $y > 0$, then the following inequality holds

$$F(xy) \geq F(x+y-1). \tag{2.8}$$

If $g(x+y-1)/g(x)$ is non-decreasing in x on $(0,\infty)$ for every $y > 0$, then (2.8) is reverses.

Proof. Set

$$G(x) = F(xy) - F(x+y-1).$$

On keeping y fixed, we have

$$\begin{aligned} G'(x) &= yF'(xy) - yF'(x+y-1) \\ &= \frac{y\ g(xy)}{f(1)}\left(\frac{g(x+y-1)}{y\ g(xy)} - 1\right). \end{aligned}$$

Set

$$H(x) = \frac{g(x+y-1)}{y\ g(xy)} - 1.$$

Since

$$G(1) = F(y) - F(y) = 0, \quad \lim_{x\to\infty} G(x) = 0, \tag{2.9}$$

then by Roll's theorem there exists a point $p \in (0,\infty)$ such that $G'(p) = 0$. Now, $H(x)$ is decreasing in x on $(0,p)$ with $H(p) = 0$, then $H(x) > 0$ on $(0,p)$. Therefore on $(0,p)$, $G'(x) > 0$, being a positive multiple of $H(x)$. This fact with (2.9) shows that $G(x)$ is increasing on $(0,p)$ vanishing at p and decreasing on (p,∞). Therefore $G(x) \geq 0$. □

3. Applications

Remark. *Theorem 1.1 follows from Theorem 2.2 by putting*

$$g(t) = t^{a-1}e^{-t}, \qquad a > 0.$$

Theorem 3.1. *Let $a, b > 0$, $a < 1$. Then the function $F(x) = B(a,b,x)/B(a,b)$ satisfy the inequality*

$$F(x)F(y) \leq F(x+y), \qquad x, y > 0. \tag{3.1}$$

Proof. The proof follows from Theorem 2.2 by putting

$$f(x) = \int_x^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt.$$

In fact,

$$\frac{g(x+y)}{g(x)} = \left(1+\frac{y}{x}\right)^{a-1}\left(1+\frac{y}{1+x}\right)^{-a-b},$$

is non-decreasing, being the product of two non-decreasing functions. □

Theorem 3.2. *Let $a > 0$, $x \geq 0$, $y > 1$. Then the function $F(x) = \Gamma(a,x)/\Gamma(a)$ satisfy the inequality (2.2). If $y \in (0,1)$, then (2.2) reverses.*

Proof. The proof follows from theorem 2.1, by putting

$$f(x) = \int_x^\infty t^{a-1}e^{-t}dt.$$

We have

$$\frac{g(xy)}{g(x)} = y^{a-1}e^{-x(y-1)},$$

which is non-decreasing for $y > 1$, and non-increasing for $0 < y < 1$. □

Theorem 3.3. *Let* $x \geq 0$, $a, b > 0$, $y > 1$. *Then the function* $F(x) = B(a,b,x)/B(a,b)$ *satisfy the inequality (2.2). If* $y \in (0,1)$, *then (2.2) reverses.*

Proof. The proof follows from theorem 2.1, by putting

$$f(x) = \int_x^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt, \qquad x \geq 0.$$

In fact

$$\frac{f'(xy)}{f'(x)} = y^{a-1} \left(\frac{1+x}{1+xy} \right)^{a+b},$$

which is non-increasing in x as

$$\left(\frac{1+x}{1+xy} \right)' = \frac{1-y}{(1+xy)^2} \leq 0, \quad \text{for} \quad y > 1.$$

□

Theorem 3.4. *Let* $x \geq 0$, $a > 0$. *Then the function* $F(x) = \Gamma(a,x)/\Gamma(a)$ *satisfy the inequality (2.8) provided* $a, y \in (1,\infty)$ *or* $a, y \in (0,1)$. *If* $a \in (0,1)$, $y \in (1,\infty)$ *or* $a \in (1,\infty)$, $y \in (0,1)$ *then (2.8) reverses.*

Proof. The proof follows from theorem 2.3, by putting

$$F(x) = \int_x^\infty t^{a-1} e^{-t} dt, \qquad x \geq 0.$$

We have

$$\frac{g(x+y-1)}{g(x)} = e^{1-y} \left(1 + \frac{y-1}{x} \right)^{a-1},$$

which implies that $\dfrac{g(x+y-1)}{g(x)}$ is non-increasing whenever

$$a, y \in (1,\infty) \, or \, a, y \in (0,1)$$

and it is non-decreasing whenever

$$a \in (0,1), y \in (1,\infty) \, or \, a \in (1,\infty), y \in (0,1).$$

□

Theorem 3.5. *Let* $b > 0$, $x \geq 0$. *Then the function* $F(x) = B(a,b,x)/B(a,b)$ *satisfy the inequality (2.8). If* $0 < y < 1$, $a > 1$. *If* $y > 1$, $0 < a < 1$, *the inequality (2.8) reverses.*

Proof. The proof follows from theorem 2.3 by putting

$$f(x) = \int_x^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt, \qquad x \geq 0.$$

We have

$$\frac{g(x+y-1)}{g(xy)} = y^{1-a} \left(1 + \frac{y-1}{x} \right)^{a-1} \left(\frac{1+xy}{x+y} \right)^{a+b},$$

$$\left(\frac{1+xy}{x+y} \right)' = \frac{y^2-1}{(x+y)^2},$$

which shows that $\dfrac{g(x+y-1)}{g(xy)}$ is non-increasing for $0 < y < 1$, $a > 1$ and it is non-decreasing for $y > 1$, $a < 1$. $\square$

Acknowledgments. The authors would like to thank the anonymous referee for his comments that helped us improve this article.

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