Journal of Algebra and Computational Applications
ISSN: 2217-6764, URL: www.ilirias.com/jaca
Volume 9 Issue 1(2020), Pages 1-8.

A DIFFERENT CHARACTERIZATION OF THE UNIT GROUP $\mathcal{U}_1(\mathbb{Z}[C_n \times C_6])$

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Abstract. Description of unit group in integral group ring of a given finite group as a subgroup of finite index is a classical problem. There are many kind of techniques for determining units. In this study, the unit group of the integral group ring of the group

$$C_n^* := C_n \times C_6 = \langle a, x : a^n = x^6 = 1, ax = xa \rangle$$

is established based on semisimplicity. At the end, we shall explicitly investigate a set of the unit generators for $n = 4$. This study introduces another way to get results in [6].

1. Introduction

Let $\mathbb{Z}G$ denote the integral group ring of a finite abelian group G with the coefficients from the ring of integers. Let $\mathcal{U}(\mathbb{Z}G)$ be the group of units in $\mathbb{Z}G$. Explicit construction of units in integral group ring is still an open problem for a specific finite abelian group. Higman [9] introduced a mostly used result as follows:

Theorem 1.1. *Let G be a finite abelian group. Then $\mathcal{U}(\mathbb{Z}G) = \pm G \times F$ where F is a torsion-free abelian group.*

The rank of torsion free part of unit group is determined by Ayoub and Ayoub [5].

Theorem 1.2. *The rank of torsion free part of unit group in the integral group ring of a finite abelian group G is*

$$\rho = \frac{1}{2}(|G| + n_2 + 1 - 2l)$$

where $|G|$ is the order of G, n_2 is the number of elements of order 2 in G and l is the number of all distinct cyclic subgroups of G.

Exemplarily, the structure of the unit groups of $\mathbb{Z}C_5$ and $\mathbb{Z}C_8$ were given by Karpilovsky with single torsion-free unit generator as follows [10] respectively:

$$\begin{aligned} \mathcal{U}(\mathbb{Z}C_5) = &\quad \pm C_5 \times \langle -1 + x + x^4 \rangle \\ \mathcal{U}(\mathbb{Z}C_8) = &\quad \pm C_8 \times \langle 2 + (x + x^7) - (x^3 + x^5) - x^4 \rangle \end{aligned}$$

2010 *Mathematics Subject Classification.* 16S34, 16U60.
Key words and phrases. integral group ring, unit group, unit generators.

Submitted February 15, 2019. Published March 9, 2020.
Communicated by Jacob Cimpric.

Aleev and Panina gave the structure of $\mathcal{U}(\mathbb{Z}C_7)$ and $\mathcal{U}(\mathbb{Z}C_9)$in [1] as

$$\begin{aligned}\mathcal{U}(\mathbb{Z}C_7) = &\ \pm C_5 \times \left\langle -1+x+x^6, -1+2(x^2+x^5)-(x^3+x^4)\right\rangle \\ \mathcal{U}(\mathbb{Z}C_9) = &\ \pm C_9 \times \left\langle -1-(x+x^8)-(x^2+x^7)+2(x^4+x^5), -1-(x+x^8)+(x^2+x^7)\right\rangle\end{aligned}$$

The torsion-free generator of $\mathcal{U}(\mathbb{Z}C_{12})$ was characterized by Bilgin [4] as

$$3+2(x+x^{11})+(x^2+x^{10})-(x^4+x^8)-2(x^5+x^7)-2x^6.$$

Low introduced a general structure of $\mathcal{U}(\mathbb{Z}[C_n \times C_2])$ with respect to the unit group $\mathcal{U}(\mathbb{Z}C_n)$ as follows [8]:

Theorem 1.3. *Since* $C_2 = \left\langle x : x^2 = 1\right\rangle$ *and* $C_n = \langle a : a^n = 1\rangle$*,*

$$\mathcal{U}(\mathbb{Z}[C_n \times C_2]) = \mathcal{U}(\mathbb{Z}C_n) \times \langle 1+(x-1)P : 1+2P \in \mathcal{U}(\mathbb{Z}C_n)\rangle$$

Low also gave the following applications of Theorem 1.3 in case of commutative groups [8].

Lemma 1.4. *Let* $C_5^* = C_5 \times C_2 = \langle c\rangle \times \langle x\rangle$*. Then,*

$$\mathcal{U}_1(\mathbb{Z}C_5^*) = \langle 1+(x-1)P\rangle \times \langle v\rangle \times C_5^*$$

where $P = -3-c+3c^2+3c^3-c^4$ *and* $v = (c+1)^2 - \hat{c}$

Lemma 1.5. *Let* $C_8^* = C_8 \times C_2 = \langle c\rangle \times \langle x\rangle$*. Then,*

$$\mathcal{U}_1(\mathbb{Z}C_8^*) = \langle 1+(x-1)P\rangle \times \langle v\rangle \times C_8^*$$

where $P = -4-3c+3c^3+4c^4+3c^5-3c^7$ *and* $v = 2+c-c^3-c^4-c^5+c^7$

He also introduced the another type of proofs of theorems 1.6, 1.7 and 1.8 [8].

Theorem 1.6. *In* $\mathcal{U}_1(\mathbb{Z}D_{12})$*,* D_{12} *has a torsion free normal complement which is a semi-direct product of a free group of rank 5 by a free group of rank 3. Moreover, this normal complement is generated by bicyclic units.*

Theorem 1.7. *In* $\mathcal{U}_1(\mathbb{Z}[D_8 \times C_2])$*,* $D_8 \times C_2$ *has a torsion-free normal complement which is a semi-direct product of a free group of rank 9 by a free group of rank 3.*

Theorem 1.8. *Let* $P = \left\langle a, b : a^4 = b^4 = 1, [b,a] = a^2\right\rangle$ *be the group of order 16. Then,*

$$\mathcal{U}(\mathbb{Z}[P \times C_2]) = \pm[F_{65} \rtimes F_9] \rtimes (P \times C_2)$$

where F_i *denotes a free group of rank* i*.*

In [3], Kelebek obtained a structure of the unit group $\mathcal{U}(\mathbb{Z}[C_n \times K_4])$ as follows

Theorem 1.9. $\mathcal{U}_1(\mathbb{Z}[C_n \times K_4]) = \mathcal{U}_1(\mathbb{Z}C_n) \times (1+K^x) \times (1+K^y) \times (1+K^{xy})$ *where*
$1+K^x = \{1+(x-1)P : 1-2P \in \mathcal{U}_1(\mathbb{Z}C_n)\}$
$1+K^y = \{1+(y-1)P : 1-2P \in \mathcal{U}_1(\mathbb{Z}C_n)\}$
$1+K^{xy} = \{1+(x-1)(y-1)P : 1+4P \in \mathcal{U}_1(\mathbb{Z}C_n)\}$

In [6], Kusmus has constructed the structure of unit group of the integral group ring $\mathbb{Z}[C_n \times C_6]$ for $C_n = \langle a : a^n = 1\rangle$ and $C_6 = \left\langle x : x^6 = 1\right\rangle$ by extending some group epimorphisms to integral group ring homomorphisms. In the next section, we shall introduce a different way of describing subgroups which have torsion-free generators of the same unit group in [6] by using Pierce Decomposition as a main result by $\mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3 \times \mathcal{U}_4 \leq \mathcal{U}(\mathbb{Z}C_n^*)$ where

$$\begin{aligned}
\mathcal{U}_1 &= \langle 1 + P.\widehat{x} : 1 + 6P \in \mathcal{U}(\mathbb{Z}C_n) \rangle \\
\mathcal{U}_2 &= \left\langle 1 + P.\widehat{-x} : 1 + 6P \in \mathcal{U}(\mathbb{Z}C_n) \right\rangle \\
\mathcal{U}_3 &= \left\langle 1 + P\alpha + Q\beta : 1 + 6[2P^2 + 2Q^2 + 2PQ + P + Q] \in \mathcal{U}(\mathbb{Z}C_n) \right\rangle \\
\mathcal{U}_4 &= \left\langle 1 + P\gamma + Q\delta : 1 + 6[2P^2 + 2Q^2 + 2PQ + P + Q] \in \mathcal{U}(\mathbb{Z}C_n) \right\rangle
\end{aligned}$$

such that

$$\begin{aligned}
\alpha &= (1 + x^3)(1 - x^2), \quad & \gamma &= (1 - x^3)(1 - x^2) \\
\beta &= (1 + x^3)(1 - x^4), \quad & \delta &= (1 - x^3)(1 - x^4)
\end{aligned}$$

2. Decomposition of $\mathcal{U}(\mathbb{Z}[C_n \times C_6])$

In this section, some remarkable objects from the theory of modules are given. In a fruitful way, some exact sequences and split extensions are constructed over the ideals of the integral group ring $\mathbb{Z}[C_n \times C_6]$. If e and f are central idempotent elements in a ring R, we know that $ef, (1-e)f, (1-f)e$ and $(1-e)(1-f)$ are also idempotent elements. Moreover, $\{ef, (1-e)f, (1-f)e, (1-e)(1-f)\}$ is a complete set of orthogonal idempotent elements of R. We also know that a commutative ring R with unity can be split into its ideals by a complete set of orthogonal idempotent elements as follows:

$$R = Ref \oplus Re(1-f) \oplus R(1-e)f \oplus R(1-e)(1-f) \tag{2.1}$$

Now, we are ready to apply this result to the integral group ring $\mathbb{Z}[C_n \times C_6]$ by using a complete set of orthogonal idempotent elements of $\mathbb{Q}C_6$. Let $C_6 = \left\langle x : x^6 = 1 \right\rangle$. Then, we can produce two orthogonal idempotent elements of $\mathbb{Q}C_6$ as $e = \frac{\widehat{x^2}}{3} = \frac{1+x^2+x^4}{3}$ and $f = \frac{\widehat{x^3}}{2} = \frac{1+x^3}{2}$. Thus, we get the following orthogonal idempotent elements:

$$\begin{aligned}
&e_1 = ef = \tfrac{1+x+x^2+x^3+x^4+x^5}{6} = \tfrac{\widehat{x}}{6} \\
&e_2 = (1-f)e = \tfrac{1-x+x^2-x^3+x^4-x^5}{6} = \tfrac{\widehat{-x}}{6} \\
&e_3 = (1-e)f = \tfrac{(1+x^3)(2-x^2-x^4)}{6} = \tfrac{2-x-x^2+2x^3-x^4-x^5}{6} \\
&e_4 = (1-e)(1-f) = \tfrac{(1-x^3)(2-x^2-x^4)}{6} = \tfrac{2+x-x^2-2x^3-x^4+x^5}{6}
\end{aligned}$$

As an application of (2.1), since $6\mathbb{Z}C_n^*$ is a subring of $\mathbb{Z}C_n^*$, we can write

$$6\mathbb{Z}C_n^* = \mathbb{Z}C_n^*\widehat{x} \oplus \mathbb{Z}C_n^*\widehat{-x} \oplus \mathbb{Z}C_n^*(1+x^3)(2-x^2-x^4) \oplus \mathbb{Z}C_n^*(1-x^3)(2-x^2-x^4) \tag{2.2}$$

Let us denote these components by

$$\begin{aligned}
I_1 &:= \mathbb{Z}C_n^*\widehat{x} \\
I_2 &:= \mathbb{Z}C_n^*\widehat{-x} \\
I_3 &:= \mathbb{Z}C_n^*(1 + x^3)(2 - x^2 - x^4) \\
I_4 &:= \mathbb{Z}C_n^*(1 - x^3)(2 - x^2 - x^4)
\end{aligned}$$

This equality (2.2) yields the following at the group level:

$$\mathcal{U}(\mathbb{Z}C_n^*) \supset \mathcal{U}(1 + I_1) \times \mathcal{U}(1 + I_2) \times \mathcal{U}(1 + I_3) \times \mathcal{U}(1 + I_4)$$

To observe the last inclusion, it is easy to check that

$$\begin{aligned}
\rho : \quad \mathbb{Z}C_n^* \quad &\longrightarrow \quad \mathbb{Z}_6C_n^* \\
\textstyle\sum_{j=0}^{5} P_j x^j \quad &\mapsto \quad \textstyle\sum_{j=0}^{5} \overline{P}_j x^j
\end{aligned}$$

is an epimorphism with $Ker(\rho) = 6\mathbb{Z}C_n^*$. Thus, a short exact sequence can be constructed as

$$6\mathbb{Z}C_n^* \xrightarrow{\iota} \mathbb{Z}C_n^* \xrightarrow{\rho} \mathbb{Z}_6 C_n^*$$

This short exact sequence can be moved to the unit group level by the restriction:

$$\begin{array}{rccc} \rho: & \mathcal{U}(\mathbb{Z}C_n^*) & \longrightarrow & \mathcal{U}(\mathbb{Z}_6 C_n^*) \\ & \sum_{j=0}^{5} P_j x^j & \mapsto & \sum_{j=0}^{5} \overline{P}_j x^j \end{array}$$

as

$$\mathcal{U}(1+6\mathbb{Z}C_n^*) \xrightarrow{\iota} \mathcal{U}(\mathbb{Z}C_n^*) \xrightarrow{\rho} \mathcal{U}(\mathbb{Z}_6 C_n^*)$$

Because $Ker\rho \supset I_k$ for all $k = 1,2,3,4$, $\epsilon(I_k) = 0$ and we investigate units which are lifted from the sets $1 + I_k$. We now expose the properties of ideals I_k for $k = 1,2,3,4$ in details.

Proposition 2.1. $I_1 = \mathbb{Z}C_n(\widehat{x})$.

Proof. Let $P = \sum_{j=0}^{5} P_j x^j \in \mathbb{Z}C_n^*$, then

$$P\widehat{x} = (\sum_{j=0}^{5} P_j x^j)\widehat{x} = (\sum_{j=0}^{5} P_j)\widehat{x} \in \mathbb{Z}C_n(\widehat{x})$$

On the other hand, an element $Q \in \mathbb{Z}C_n$ can be written as the sum of 6 elements as $Q = \sum_{j=0}^{5} P_j^{'}$. Also, we know that $\widehat{x} = x.\widehat{x} = x^2.\widehat{x} = x^3.\widehat{x} = x^4.\widehat{x} = x^5.\widehat{x}$. Thus, we can write that $Q\widehat{x} = (\sum_{j=0}^{5} P_j)(\widehat{x}) = (\sum_{j=0}^{5} P_j x^j)\widehat{x} \in \mathbb{Z}C_n^*(\widehat{x})$. □

Similarly, we can easily check without giving the proof that

$$I_2 = \mathbb{Z}C_n^*(\widehat{-x}) = \mathbb{Z}C_n(\widehat{-x})$$

Proposition 2.2.

$$\begin{array}{l} I_3 = (1+x^3)(1-x^2)\mathbb{Z}C_n \oplus (1+x^3)(1-x^4)\mathbb{Z}C_n \\ I_4 = (1-x^3)(1-x^2)\mathbb{Z}C_n \oplus (1-x^3)(1-x^4)\mathbb{Z}C_n \end{array}$$

Proof.

$$\begin{array}{rcl} I_3 & = & \{(\sum_{i=0}^{5} P_i x^i)(1+x^3)[3-(1+x^2+x^4)] : P_i \in \mathbb{Z}C_n\} \\ & = & \{3[(P_0+P_3)(1+x^3)+(P_1+P_4)(x+x^4)+(P_2+P_5)(x^2+x^5)] - \\ & & (\sum P_i)(1+x^2+x^4) : P_i \in \mathbb{Z}C_n\} \\ & = & \{Q_0+Q_1x+Q_2x^2+Q_0x^3+Q_1x^4+Q_2x^5 : Q_i \in \mathbb{Z}C_n, Q_0+Q_1+Q_2=0\} \\ & = & \{(-Q_1-Q_2)+Q_1x+Q_2x^2+(-Q_1-Q_2)x^3+Q_1x^4+Q_2x^5 : Q_i \in \mathbb{Z}C_n\} \\ & = & \{Q_1(-1+x-x^3+x^4)+Q_2(-1+x^2-x^3+x^5) : Q_i \in \mathbb{Z}C_n\} \\ & = & \{Q_2(1+x^3)(1-x^2)+Q_1(1+x^3)(x^4-1) : Q_i \in \mathbb{Z}C_n\} \\ & = & (1+x^3)(1-x^2)\mathbb{Z}C_n \oplus (1+x^3)(1-x^4)\mathbb{Z}C_n \square \end{array}$$

The same procedure can be applied for the proof of the second equality in the proposition. □

Now, let us characterize the unit subgroups $\mathcal{U}(1+I_k)$ for $k = 1,2,3,4$ explicitly.

Lemma 2.3.

$$\begin{array}{l} \mathcal{U}(1+I_1) = \{1+P.(\widehat{x}) : 1+6P \in \mathcal{U}(\mathbb{Z}C_n)\} \\ \mathcal{U}(1+I_2) = \{1+P.(\widehat{-x}) : 1+6P \in \mathcal{U}(\mathbb{Z}C_n)\} \end{array}$$

Proof.
Keeping in mind the fact that $\mathcal{U}(1+I_k) = (1+I_k) \cap \mathcal{U}(\mathbb{Z}C_n^*)$, let us consider an element $1 + P(\widehat{x}) \in 1 + I_1$. Then,

$$\begin{aligned}
u = 1 + P(\widehat{x}) \text{ is a unit} &\iff \exists v = 1 + Q(\widehat{x}) : uv = 1 \\
&\iff (1 + P(\widehat{x}))(1 + Q(\widehat{x})) = 1 \\
&\iff 1 + \widehat{x}[P + Q + 6PQ] = 1 \\
&\iff P + Q + 6PQ = 0 \\
&\iff 1 + 6P + 6Q + 36PQ = 1 \\
&\iff (1 + 6P)(1 + 6Q) = 1 \\
&\iff 1 + 6P \in \mathcal{U}(\mathbb{Z}C_n) \ \square
\end{aligned}$$

The other one is similarly proved.

Lemma 2.4.

$$\begin{aligned}
\mathcal{U}(1+I_3) &= \{1 + (1+x^3)(1-x^2)P + (1+x^3)(1-x^4)Q : P, Q \in \mathbb{Z}C_n\} \\
\mathcal{U}(1+I_4) &= \{1 + (1-x^3)(1-x^2)P + (1-x^3)(1-x^4)Q : P, Q \in \mathbb{Z}C_n\}
\end{aligned}$$

such that $1 + 6[2P^2 + 2Q^2 + 2PQ + P + Q] \in \mathcal{U}(\mathbb{Z}C_n)$.

Proof.
Let $\alpha = (1+x^3)(1-x^2)$ and $\beta = (1+x^3)(1-x^4)$. Then, we can get the following multiplication table:

$\cdot$	α	β
α	$4\alpha - 2\beta$	$2\alpha + 2\beta$
β	$2\alpha + 2\beta$	$-2\alpha + 4\beta$

Let us search which types of elements of the form $1 + \alpha P + \beta Q$ are units. $u = 1 + \alpha P + \beta Q$ is a unit

$$\iff \exists v = 1 + R\alpha + S\beta : uv = 1$$

$$\iff (1 + \alpha P + \beta Q)(1 + R\alpha + S\beta) = 1$$

$$\iff \alpha(P+R) + \beta(Q+S) + 2(\alpha+\beta)(PS+QR) + (-2\alpha+4\beta)QS + (4\alpha-2\beta)PR = 0$$

$$\iff \alpha(P + R + 2PS + 2QR - 2QS + 4PR) + \beta(Q + S + 2PS + 2QR + 4QS - 2PR) = 0$$

$$\iff P + R + 2PS + 2QR - 2QS + 4PR = 0 \,\text{and}\, Q + S + 2PS + 2QR + 4QS - 2PR = 0$$

$$\iff \begin{bmatrix} 1+4P+2Q & 2P-2Q \\ -(2P-2Q) & 1+2P+4Q \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} -P \\ -Q \end{bmatrix} \text{has a unique solution in } \mathbb{Z}C_n$$

$$\iff 1 + 6[2P^2 + 2Q^2 + 2PQ + P + Q] \in \mathcal{U}(\mathbb{Z}C_n)$$

$\mathcal{U}(1+I_4)$ can easily be obtained as previous one similarly. To sum up, we can state the main result of this section as follows:

Corollary 2.5. $\mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3 \times \mathcal{U}_4 \leq \mathcal{U}(\mathbb{Z}C_n^*)$ *where*

$$\begin{aligned}
\mathcal{U}_1 &= \langle 1 + P.\widehat{x} : 1 + 6P \in \mathcal{U}(\mathbb{Z}C_n) \rangle \\
\mathcal{U}_2 &= \left\langle 1 + P.\widehat{-x} : 1 + 6P \in \mathcal{U}(\mathbb{Z}C_n) \right\rangle \\
\mathcal{U}_3 &= \left\langle 1 + P\alpha + Q\beta : 1 + 6[2P^2 + 2Q^2 + 2PQ + P + Q] \in \mathcal{U}(\mathbb{Z}C_n) \right\rangle \\
\mathcal{U}_4 &= \left\langle 1 + P\gamma + Q\delta : 1 + 6[2P^2 + 2Q^2 + 2PQ + P + Q] \in \mathcal{U}(\mathbb{Z}C_n) \right\rangle
\end{aligned}$$

such that

$$\begin{array}{rclrcl}\alpha & = & (1+x^3)(1-x^2), & \gamma & = & (1-x^3)(1-x^2)\\ \beta & = & (1+x^3)(1-x^4), & \delta & = & (1-x^3)(1-x^4)\end{array}$$

3. Torsion-free Unit Generators for $n = 4$

In this section, we apply Corollary 2.1 to a non-cyclic abelian group of order 24. In other words, we deal with $\mathcal{U}_1$,$\mathcal{U}_2$,$\mathcal{U}_3$,$\mathcal{U}_4$ for integral group ring $\mathbb{Z}[C_4 \times C_6]$ since $C_4 = \langle a : a^4 = 1\rangle$ and $C_6 = \langle x : x^6 = 1\rangle$.At first, we can easily see that $\mathcal{U}_1$ and $\mathcal{U}_2$ are trivial because of $\mathcal{U}(\mathbb{Z}C_4) = \pm C_4$. Then, characterization of $\mathcal{U}_3$ and $\mathcal{U}_4$ completes the framework. However, invention of elements P and Q satisfying

$$1 + 6[2P^2 + 2Q^2 + 2PQ + P + Q] \in \mathcal{U}(\mathbb{Z}C_n)$$

is a hard problem. Hence, we consider the following assumption:

$$2P^2 + 2Q^2 + 2PQ + P + Q = 0 \Longrightarrow [3(P + Q) + 1]^2 + 3(P - Q)^2 = 1 \qquad (3.1)$$

Now, let us consider (3.1) in $\mathbb{Z}C_4$. Let $P = x_0 + x_1a + x_2a^2 + x_3a^3 \in \mathbb{Z}C_4$ and $Q = y_0 + y_1a + y_2a^2 + y_3a^3 \in \mathbb{Z}C_4$. Since ε is the augmentation map, we denote

$$\varepsilon(P) = x_0 + x_1 + x_2 + x_3 = m \text{ and } \varepsilon(Q) = y_0 + y_1 + y_2 + y_3 = n$$

In this case, the integer solution of the equation $[3(m + n) + 1]^2 + 3(m - n)^2 = 1$ is only $m = n = 0$. That is

$$\begin{array}{c} x_0 + x_1 + x_2 + x_3 = 0 \\ y_0 + y_1 + y_2 + y_3 = 0 \end{array} \qquad (3.2)$$

Besides, consider the following group homomorphism:

$$\begin{array}{rccc}\psi : & C_4 & \longrightarrow & \mathbb{C} \\ & a^j & \mapsto & i^j\end{array}$$

If we linearly extend ψ over integers, we obtain

$$\begin{array}{rccc}\bar{\psi} : & \mathbb{Z}C_4 & \longrightarrow & \mathbb{Z}[i] \\ & P = \sum_{j=0}^{3} x_ja^j & \mapsto & (x_0 - x_2) + (x_1 - x_3)i \\ & Q = \sum_{j=0}^{3} y_ja^j & \mapsto & (y_0 - y_2) + (y_1 - y_3)i\end{array}$$

Therefore, the equation

$$[3(\bar{\psi}(P) + \bar{\psi}(Q)) + 1]^2 + 3[\bar{\psi}(P) - \bar{\psi}(Q)]^2 = 1$$

yields

$$(3[(x_0-x_2+y_0-y_2)+(x_1-x_3+y_1-y_3)i]+1)^2+3[(x_0-x_2-y_0+y_2)+(x_1-x_3-y_1+y_3)i]^2 = 1$$

Notice that this equation has nonzero integer solutions if and only if

$$\begin{array}{c} x_1 - x_3 + y_1 - y_3 = 0 \\ x_0 - x_2 - y_0 + y_2 = 0 \end{array} \qquad (3.3)$$

In addition, consider the ring homomorphism:

$$\begin{array}{rccc}\vartheta : & \mathbb{Z}C_4 & \longrightarrow & \mathbb{Z}\langle a^2\rangle \\ & P = \sum_{i=0}^{3} x_ia^i & \mapsto & (x_0 + x_2) + (x_1 + x_3)a^2 \\ & Q = \sum_{i=0}^{3} y_ia^i & \mapsto & (y_0 + y_2) + (y_1 + y_3)a^2\end{array}$$

Also, let $\vartheta(P) = P^{'}$ and $\vartheta(Q) = Q^{'}$. Then,

$$\bar{\psi}(P^{'}) = \bar{\psi}[(x_0 + x_2) + (x_1 + x_3)a^2] = x_0 + x_2 - x_1 - x_3 = X$$

and

$$\bar{\psi}(Q^{'}) = \bar{\psi}[(y_0 + y_2) + (y_1 + y_3)a^2] = y_0 + y_2 - y_1 - y_3 = Y$$

Thus, if we substitute X and Y into the equation (3.1), we conclude that $X = Y = 0$. This means

$$\begin{array}{c} x_0 + x_2 - (x_1 + x_3) = 0 \\ y_0 + y_2 - (y_1 + y_3) = 0 \end{array} \tag{3.4}$$

Hence, by the equations (3.2) and (3.4), we attain

$$\begin{array}{cc} x_0 + x_2 = 0 \ , & y_0 + y_2 = 0 \\ x_1 + x_3 = 0 \ , & y_1 + y_3 = 0 \end{array} \tag{3.5}$$

The equation (3.5) shows that $(3[x_0 - x_2 + y_0 - y_2] + 1)^2 + 3[(x_1 - x_3 - y_1 + y_3)i]^2 = 1$. From the equations (3.2)-(3.5) and [7],we obtain

$$\begin{array}{rcl} [3(2x_0 + 2x_0) + 1]^2 - 3(2x_1 + 2x_1)^2 = 1 & \Longrightarrow & (12x_0 + 1)^2 - 48x_1^2 = 1 \\ & \Longrightarrow & (12x_0 + 1) + 4x_1\sqrt{3} \in \mathcal{U}(\mathbb{Z}[\sqrt{3}]) \\ & \Longrightarrow & (12x_0 + 1) + 4x_1\sqrt{3} = (2 + \sqrt{3})^k \\ & \Longrightarrow & 12x_0 + 1 = 97 \, , \, 4x_1 = 56 \\ & \Longrightarrow & x_0 = 8 \, , \, x_1 = 14 \\ & \Longrightarrow & P = 8 + 14a - 8a^2 - 14a^3 \end{array}$$

As $x_0 = y_0$ and $x_1 = -y_1$, we get

$$Q = 8 - 14a - 8a^2 + 14a^3$$

In consequent, we can state the main result in this section as follows:

Corollary 3.1. *Let* $C_4^* := C_4 \times C_6 = \langle a, x : a^4 = x^6 = 1, ax = xa \rangle$ *. Then,*

$$\mathcal{U}(\mathbb{Z}C_4^*) = \pm C_4^* \times \langle u, v \rangle$$

where the torsion-free unit generators are

$$u = 1 + (1 + x^3)(1 - x^2)[8 + 14a - 8a^2 - 14a^3] + (1 + x^3)(1 - x^4)[8 - 14a - 8a^2 + 14a^3]$$
$$v = 1 + (1 - x^3)(1 - x^2)[8 + 14a - 8a^2 - 14a^3] + (1 - x^3)(1 - x^4)[8 - 14a - 8a^2 + 14a^3]$$

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] R. Zh. Aleev and G. A. Panina, The Units Cyclic Groups of Orders 7 and 9, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat., **11** (2000), 80-83.

[2] I.G. Kelebek and T. Bilgin, Computing The Rank of $\mathcal{U}(\mathbb{Z}A)$, Int. J. Algebra, **7** (2013), 145-156.

[3] I.G. Kelebek and T. Bilgin, Characterization of $\mathcal{U}_1(\mathbb{Z}[C_n \times K_4])$, Eur. J. Pure Appl. Math., **7** (2014), 462-471.

[4] T. Bilgin, Characterization of $\mathcal{U}_1(\mathbb{Z}C_{12})$, Int. J. Pure Appl. Math., **14** (2004), 531-535.

[5] R.G. Ayoub and C. Ayoub, *On the Group Ring of A Finite Abelian Group*, Bull. Aust. Math. **1** (1969), 245-261.

[6] O. Kusmus, On the Units of Integral Group Ring of $C_n \times C_6$, Algebra Discrete Math.,**20** (2015), 142-151.

[7] Dickson L. E., *History of The Theory of Numbers, Volume II: Diophantine Analysis*, Chelsea Pub. Comp.,New York,(1966).

[8] R. M. Low, *On The Units of The Integral Group Ring* $\mathbb{Z}[G \times C_p]$, J. Algebra Appl., **7** (2008), 393-403.

[9] G. Higman, *The Units of Group-Rings* , Proc. London Math. Soc., **46** (1940), 231-248.

[10] G. Karpilovsky, *Unit Groups of Group Rings*, John Wiley and Sons Inc., New York, (1989).

[11] Sehgal S.K., *Units in Integral Group Rings*,Marcel Dekker,New York,Besel,(1993).

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