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BOUNDED ARCHIMEDEAN F-RINGS

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ABSTRACT. In this paper we study the category of bounded Archimedean lattice-ordered rings which we denote baf and follow the authors notation in [4] by denoting the category of bounded Archimedean ℓ-algebras **bal**. In particular, we present a ring-theoretic parallel of the comparison between vector lattices and ℓ-groups by proving some similarities and differences between **baf** and **bal**.

1. INTRODUCTION

Ordered Algebraic Structures form a bridge between algebra and topology, which led mathematicians to study them. A pivot structure in this area is the rings of continuous functions over the real numbers, $C(X,\mathbb{R})$; topological properties of X can be translated into algebraic behavior of $C(X,\mathbb{R})$. It is well-known that $C(X,\mathbb{R})$ admits an order, namely $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$, this makes $C(X,\mathbb{R})$ a commutative bounded and Archimedean ℓ-algebra.

In year 1943 Gelfand Neymark showed that there is a duality between the category of compact Hausdorff spaces and Stone rigns. The later are bounded archimedean ℓ-algebras with a norm that makes it a Banach Algebra [4]. This duality gave rise to a lot of work in this area. The category of bounded Archimedean f-rings is our main interest in this paper. we will denote it by **baf**. **bal** will denote the category of bounded Archimedean ℓ-algebras. In section 2 we present the necessary machinery needed in order to characterize the **bal** reflector presented in section 3 and we show that **bal** is a reflective subcategory of **baf**. In the fourth section we investigate some special subcategories of **baf** and **bal** and prove some similarities and crucial differences between the two.

2. PRELIMINARIES

All rings we consider are assumed to be commutative with a unit and all homomorphisms are unit preserving. We will start with few important definitions (see [5, Chapters 13-17]).

Definition 2.1.

(1) Let A be a ring endowed with a partial order $\leq$. Then A is said to be a lattice ordered ring, abbreviated ℓ-ring if :

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- $(A, \leq)$ is a lattice.
- If $a \leq b$, then $a+c \leq b+c$ for all c.
- If $a \geq 0, b \geq 0$, then $ab \geq 0$.

(2) An ℓ-ring A is said to be an f-ring if for all $a,b,c \in A$ with $a \wedge b = 0$ and $c \geq 0$ we have $ac \wedge b = 0$.
(3) An ℓ-ring A possesses the Archimedean property, if whenever $a,b \in A$ satisfy $na \leq b$ for all $n \in \mathbb{N}$, then $a \leq 0$.
(4) An ℓ-ring is bounded if for each $a \in A$ there exists an $n \in \mathbb{N}$ such that $a \leq n.1_A$.
(5) An ℓ-ring A has the bounded inversion property if whenever $a \geq 1$, then a is a unit.
(6) A commutative ring with a unit A is called Gelfand, if whenever $a,b \in A$ such that $a+b=1$, there exist $r,s \in A$ such that $(1+ra)(1+sb)=0$.

Remark 2.2. The zero ring trivially belongs to **baf** and it is the terminal object of the category. On the other hand, $\mathbb{Z}$ is the initial object in **baf** because morphisms in **baf** are unit preserving. Most results in what is to come hold for the zero ring, so unless problematic, we will often skip the verifications in the proofs for the trivial case.

We begin with some natural examples of objects in **baf**.

Example 2.3.

(1) $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. In fact, any unital subring of $\mathbb{R}$ is an object in **baf**.
(2) A fundamental role will be played by $C(X,\mathbb{R})$. It is well known that whenever X is compact $C(X,\mathbb{R})$ endowed with component-wise order $\leq$ is a bounded Archimedean ℓ-algebra, (see [4, Example 2.5]). Moreover as each continuous function that does not vanish anywhere has a continuous inverse, $C(X,\mathbb{R})$ has bounded inversion. Furthermore, what makes $C(X,\mathbb{R})$ so important in functional analysis is that it is uniformly complete, meaning that it is complete with respect to the uniform norm given by
$$||f|| = sup\{|f(x)| : x \in X\}.$$
We will call such an algebraic structure (i.e an object in **baf** that is an ℓ-algebra and uniformly closed in the norm) a Stone ring following Banaschewski in [2].
(3) There are objects in **baf** without bounded inversion; for example $\mathbb{Z}$. For another, let X be the closed interval $[0,1]$. We say that $f \in C(X,\mathbb{R})$ is a piece-wise polynomial function with integer coefficients if there exists a partition of X, say $0 = a_0 < a_1 < \cdots < a_n = 1$ such that $f = p_i \in \mathbb{Z}[x]$, whenever $x_i \in [a_{i-1}, a_i]$. Let $PP_{\mathbb{Z}}(X)$ be the set of all piecewise polynomial functions with integer coefficients on X. Then it is a standard exercise showing $PP_{\mathbb{Z}}(X)$ is closed under the ring operations and so $PP_{\mathbb{Z}}(X)$ is an ℓ-subring of $C(X,\mathbb{R})$ and so is in **baf** and not in **bal**.
(4) $C(X,\mathbb{Z})$.
(5) $\mathbb{Z}$ localized at a prime ideal (p) is an example of a **baf** object.
(6) If $A,B \in$ **baf** then $A \times B$ with componentwise operations is also in **baf**.
(7) Any sub-ℓ-ring of an object in **baf** is also in **baf**.

Definition 2.4. Let $A \in$ **baf**. We say that an ideal I of A is an ℓ-ideal if:

(1) I is convex, i.e if a and b belong to I, then any c satisfying $a \leq c \leq b$ also belongs to I.
(2) I is closed under the lattice operations, i.e $a,b \in I \implies a \vee b \in I$ and $a,b \in I \implies a \wedge b \in I$.

A proper ℓ-ideal that is not contained in any proper ℓ-ideal other than itself will be called a *maximal* ℓ-ideal.

It can be easily shown that every maximal ℓ-ideal of $A \in$ **baf** is a prime ideal. As a consequence we get that every A/M is totally ordered. Moreover Johnstone in [25, Theorem 2.11] shows that the intersection of all maximal ℓ-ideals is (0), i.e A has no non-zero nilpotent elements, which means that A/I is Archimedean if and only if $I = \bigcap\{maximal \ell - ideals\}$.

Remark 2.5. Unlike in the category **bal**, a maximal ℓ-ideal of an object in **baf** might not be a maximal ideal. For instance, (0) is a maximal ℓ-ideal but not a maximal ideal of $\mathbb{Z}$.

Let $A \in$ **baf** and let X_A denote the set of all maximal ℓ-ideals of A. In [7, Section 4.8], Johnstone shows that X_A is a compact Hausdorff space with respect to the Zariski topology induced by $\mathrm{Spec}(A)$. As a consequence, A/M is isomorphic to a unique subring of $\mathbb{R}$, so to each element $a \in A$, we can assign a real valued function $f_a : X_A \longrightarrow \mathbb{R}$ defined by $f_a(M) = a + M$, it is an exercise showing that these are continuous functions. Hence, defining $\varphi_A : A \longrightarrow C(X_A, \mathbb{R})$ by $\varphi_A(a) = f_a$ gives and injective ℓ-ring homomorphism. We get the following theorem.

Theorem 2.6. *If $A \in$ **baf**, then X_A is a compact Hausdorff space and $\varphi_A : A \longrightarrow C(X_A, \mathbb{R})$ is a 1-1 morphism in **baf**. Conversely, if A is isomorphic to an ℓ-subring of $C(X, \mathbb{R})$, for X compact Hausdorff, then $A \in$ **baf**. Thus, an ℓ-ring A is in **baf** if and only if A is isomorphic to an ℓ-subring of $C(X, \mathbb{R})$ for some compact Hausdorff space X.*

The following propositions will be useful in the categorical discussion we have in later sections.

Proposition 2.7. *Let $A,B \in$ **baf** and $\alpha : A \longrightarrow B$ be a morphism in **baf**. Then α is monic if and only if α is one to one.*

Proof. Suppose α is monic. If A is the zero ring, then α is clearly one to one. Suppose A is nonzero and let $I = \ker(\alpha)$. It can be easily seen that $\mathbb{Z} + I$ is an ℓ-subring of A. Let $\beta : \mathbb{Z} + I \longrightarrow A$ be the identity morphism and $\gamma : \mathbb{Z} + I \longrightarrow A$ be the morphism given by $\gamma(a+i) = a$ for $a \in \mathbb{Z}$ and $i \in I$. It is clear that β is a morphism in **baf**. To see that γ is too, let $a,b \in \mathbb{Z}$ and $i, j \in I$ and suppose $a \leq b$, then :

$$\gamma((a+i)+(b+j)) = \gamma((a+b)+(i+j)) = a+b = \gamma(a+i)+\gamma(b+j)$$
$$\gamma((a+i)(b+j)) = \gamma((ab)+(aj+bi+ij)) = ab = \gamma(a+i)\gamma(b+j)$$
$$\gamma((a+i)\vee(b+j)) = \gamma(b+(a-b+i)\vee j) = b = a\vee b = \gamma(a+i)\vee\gamma(b+j),$$

since $j \leq (a-b+i)\vee j \leq i \vee j$, so $a-b+i \in I$. Similarly for the meet. So γ is a **baf**-morphism. Moreover,

$$\alpha(\beta(a+i)) = \alpha(a+i) = \alpha(a+0)+\alpha(0+i) = \alpha(a)$$

and

$$\alpha(\gamma(a+i)) = \alpha(a).$$

Thus $\alpha\circ\beta = \alpha\circ\gamma$ giving $\beta = \gamma$. Therefore, $\gamma(a+i) = \beta(a+i)$ implying $a = a+i$ and so $i = 0$ for any $a+i \in \mathbb{Z}+I$, hence $I = 0$ and α is one to one. ☐

The proposition above follows from a more general fact involving bounded Archimedean ℓ-groups, [1].

We will soon see that epimorphisms in **baf** may not be onto and so there are bimorphisms in **baf** (monic and epic morphisms) which are not isomorphisms.

In order to achieve a categorical setting, we investigate what happens to the morphisms in **baf**. Given $\alpha : A \longrightarrow B$ in **baf**, we define a map $\alpha^* : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ by $\alpha^*(P) = \alpha^{-1}(P)$ for $P \in \operatorname{Spec}(B)$. It is a well known fact that α^* is continuous. If $M \in X_B$, then $\alpha^*(M) \in X_A$, so $\alpha^* : X_B \longrightarrow X_A$ is also continuous.

Proposition 2.8. *Let $\alpha : A \longrightarrow B$ be a morphism in **baf**.*

(1) $\alpha : A \longrightarrow B$ is monic if and only if $\alpha^* : X_B \longrightarrow X_A$ is onto.
(2) $\alpha : A \longrightarrow B$ is epic if and only if $\alpha^* : X_B \longrightarrow X_A$ is one to one.
(3) $\alpha : A \longrightarrow B$ is a bimorphism if and only if $\alpha^* : X_B \longrightarrow X_A$ is a homeomorphism.

Proof. See [4] □

Corollary 2.9. $\varphi_A : A \longrightarrow C(X_A, \mathbb{R})$ *is a bimorphism.*

Proof. $X_{C(X_A,\mathbb{R})} \cong X_A$. □

From here we see that one can get epic morphisms that are not onto as A is not isomorphic to $C(X_A, \mathbb{R})$.

3. The bal Reflector

It turns out that C and X are contravariant functors between the categories of compact Hausdorff spaces and **baf**. In fact, there is a catravariant adjunction between these two categories.

Remark 3.1. For $\alpha : A \longrightarrow B$ we have $\alpha^{-1} = \alpha^* : X_B \longrightarrow X_A$ and we get the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, hook, "\varphi_A"'] & B \arrow[d, hook, "\varphi_B"] \\
C(X_A, \mathbb{R}) \arrow[r, "C(\alpha^*)"'] & C(X_B, \mathbb{R})
\end{tikzcd}$$

Proposition 3.2. ***bal*** *is a full subcategory of* ***baf***.

Proof. By abuse of notation, we will view **bal** as a subcategory of **baf** via the forgetful functor (forgetting scalar multiplication). Let us check that a **baf**-morphism between A and B in **bal** is also a **bal**-morphism. Let $\phi : A \longrightarrow B$ be a **baf**-morphism. So ϕ is an ℓ-ring homomorphism. Let us check that it is also an $\mathbb{R}$-algebra homomorphism. Let $r \in \mathbb{R}$, $a \in A$ and let $q_n = \dfrac{c_n}{d_n}$ be a sequence of rationals converging to r with $d_n > 0$. Then identifying $\mathbb{R} \subseteq A$ via $r \to r.1_A$, we have

$$\phi(ra) = \phi(r)\phi(a).$$

Hence, it is enough to show that $\phi(r) = r$. But :

$$\phi(r) = \phi(\lim_{n\to\infty} \frac{c_n}{d_n}) = \lim_{n\to\infty} \phi(\frac{c_n}{d_n}) = \lim_{n\to\infty} c_n \phi(\frac{1}{d_n}) = \lim_{n\to\infty} c_n \phi(d_n)^{-1} = \lim_{n\to\infty} \frac{c_n}{d_n} = r,$$

where the second equality holds because ϕ is continuous. □

So **bal** is a full subcategory of **baf**.

Definition 3.3. A full subcategory **D** of a category **C** is said to be reflective in **C** whenever the inclusion functor has a left adjoint. That is, if for each $C \in \mathbf{C}$ there is a $D \in \mathbf{D}$ and a **C**-morphism $r : C \longrightarrow D$, called the reflector, such that for each **C**-morphism $\alpha : C \longrightarrow D'$, where $D' \in \mathbf{D}$, there is a unique **D**-morphism $\beta : D \longrightarrow D'$ with $\beta \circ r = \alpha$, i.e, the following diagram commutes.

$$\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "r"'] & D' \\
D \arrow[ur, dotted, "\beta"'] &
\end{tikzcd}$$

Proposition 3.4. ***bal*** *is a reflective subcategory of* ***baf***.

Proof. Let $A \in \mathbf{baf}$, then $\varphi_A(A) \subseteq C(X_A, \mathbb{R})$. Let $A' = \bigcap\{C \in \mathbf{bal} \mid \varphi_A(A) \subseteq C \subseteq C(X_A, \mathbb{R})\}$. Then clearly $A' \in \mathbf{bal}$ and φ_A can be viewed as $\varphi_A : A \longrightarrow A'$. Now let $B \in \mathbf{bal}$ and $\alpha : A \longrightarrow B$ be a baf morphism. We have the following diagram where j is an isomorphism:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, hook, "\varphi_A"'] & B \arrow[dd, hook, "\varphi_B"] & \\
A' \arrow[d] & & \\
C(X_A, \mathbb{R}) \arrow[r, "C(\alpha^*)"'] & C(X_B, \mathbb{R}) & \varphi_B(B) \arrow[l, hook'] \arrow[uul, leftrightarrow, "j"']
\end{tikzcd}$$

Let us first show

$$\varphi_A(A) \subseteq (C(\alpha^*))^{-1}(\varphi_B(B)) \subseteq C(X_A, \mathbb{R}), \qquad (*)$$

Let $a \in A$, then

$$C(\alpha^*)(\varphi_A(a)) = C(\alpha^*)(f_a) = f_a \circ \alpha^*,$$

We need to show that $f_a \circ \alpha^* \in \varphi_B(B)$, i.e we need $b \in B$ with $f_a \circ \alpha^* = f_b$. For that, let $b = \alpha(a)$. As $f_{\alpha(a)} = f_a \circ \alpha^*$, we get the desired conclusion and so equation $(*)$ is shown. As $\varphi_B(B) \in \mathbf{bal}$, we have $(C(\alpha^*))^{-1}(\varphi_B(B)) \in \mathbf{bal}$ and hence $A' \subseteq (C(\alpha^*))^{-1}(\varphi_B(B))$. Thus define $\lambda := j^{-1} \circ C(\alpha^*)|_{A'}$. λ is a well-defined **bal** morphism, being a composition of two **bal** morphisms. Moreover, λ makes the following diagram commute because by the diagram above

$$\lambda \circ \varphi_A = j^{-1} \circ C(\alpha^*)|_{A'} \circ \varphi_A = \alpha,$$

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\varphi_A"'] & B \\
A' \arrow[ur, dotted, "\lambda"'] &
\end{tikzcd}$$

Furthermore, supposing there is another λ' making this diagram commute, then

$$\lambda \circ \varphi_A = \lambda' \circ \varphi_A,$$

and as φ_A is epic, we get $\lambda = \lambda'$. Thus **bal** is a reflective subcategory of **baf**. □

Corollary 3.5. *For $A \in$ **baf**, the reflector from **baf** to **bal** is given by*

$$r(A) = \bigcap\{C \in \textbf{\textit{bal}} \mid \varphi_A(A) \subseteq C \subseteq C(X_A, \mathbb{R})\}.$$

*To ease notation $r(A)$ will be denoted by $\widetilde{A}$ for every $A \in$ **baf**.*

Remark 3.6. Each reflective subcategory of **bal** is a reflective subcategory of **baf**.

Some of those are: **ubal**, the category of Stone rings, **bibal**, the category of bounded Archimedean ℓ-algebras with bounded inversion, **scbal**, the category of bounded Archimedean ℓ-algebras that are square closed. We will denote the corresponding subcategories of **baf** by **ubaf**, **bibaf** and **scbaf** respectively. The main reason we are discussing these subcategories is to illustrate where the main differences and similarities of **baf** and **bal** reside.

4. Some other reflectors and applications

The following section presents some reflective subcategories of baf and some categorical consequences. However, some well-known theorems are needed first.

Definition 4.1. A commutative ring with identity is said to be a pm-ring if each prime ideal is contained in a unique maximal ideal.

The following theorem can be found in [6, Theorem 4.1].

Theorem 4.2. *Let A be a commutative ring with identity. Then A is Gelfand if and only if A is a pm-ring.*

We write $J(A)$ for the Jacobson radical of a ring A.

Theorem 4.3. *Let A be a commutative ring with identity. If* $\operatorname{Max}(A)$ *is Hausdorff then* $A/J(A)$ *is Gelfand.*

Proof. [9]. □

Corollary 4.4. *If A is a domain, then A is Gelfand iff A is local.*

Proof. As A is Gelfand, A is a pm-ring by Theorem 4.2 and as A is a domain, (0) is a prime ideal. So (0) is contained in a unique maximal ideal and so A is local. Conversely, if A is local, A is a pm-ring and so A is Gelfand. □

Proposition 4.5. *Consider the following statements in **baf**.*

(1) *A has bounded inversion.*
(2) *Max(A)= X_A*
(3) *A is Gelfand.*
(4) *Max(A) is Hausdorff.*

Then $(1) \iff (2)$ *and* $(2) \implies (3) \implies (4)$.

Proof. $(1) \implies (2)$ Suppose A has bounded inversion and let $M \in Max(A)$. We will show that M is an ℓ-ideal. Suppose $b \notin M$ and $|b| \leq |a|$. We show $a \notin M$. There exists $x \in A$ with $bx + m = 1$ for some $m \in M$. So $b^2x^2 + m' = 1$ where $m' = 2bxm + m^2 \in M$, giving $a^2x^2 + m' \geq 1$ since $b^2 \leq a^2$. Hence, $a^2x^2 + m'$ is a unit in A. So there exists $u \in A$ such that $(a^2x^2 + m')u = 1$ implying $a(ax^2u) + M = 1 + M$ which means $a \notin M$. Thus M is an ℓ-ideal and it is not contained in any proper ℓ-ideal, so $M \in X_A$. Conversely, Let $M \in X_A$. Then M is prime, so A/M is a totally ordered Archimedean integral domain. We show A/M is a field. Let $a \in A$ and without loss of generality suppose $a + M > 0 + M$. As A/M

is Archimedean, there exists $n \in \mathbb{N}$ with $n(a+M) > 1+M$, so by definition of the order on A/M, there exists $a' \in na+M$ with $a' \geq 1$. Therefore, $a'+M = na+M$ and a' is invertible. So there exists $u \in A$ with $a'u = 1$. Equivalently, there exists $u \in A$, with $a'u+M = 1+M$, so $una+M = 1+M$. But $una+M = (u+M)(na+M)$ and therefore $na+M$ is invertible and A/M is a field. So $M \in Max(A)$.

(2) $\implies$ (1). Suppose $Max(A) = X_A$ and let $a \in A$ with $a \geq 1$. Then a is in no proper ℓ-ideal of A (otherwise, $a \geq 1 \geq 0$ and 1 is in the ℓ-ideal). So a is in no maximal ideal and so a is a unit. So A has bounded inversion.

(2) $\implies$ (3), (4) If $Max(A) = X_A$, then as X_A is Hausdorff by Proposition **??**, $Max(A)$ is also Hausdorff and so (3) is valid. By Theorem 4.3 $A/J(A)$ is Gelfand, but $J(A) = \bigcap X_A = 0$, so A is Gelfand.

(3) $\implies$ (4) Suppose $M \neq N \in Max(A)$. Then $M+N = A$, so there exist $a \in M$, $b \in N$ with $a+b = 1$. As A is Gelfand, there exist r and s such that $(1+ra)(1+sb) = 0$. Let $D = Z(1+ra)$ and $E = Z(1+sb)$. If $M \in D$ then $1+ra \in M$ giving $1 \in M$, a contradiction. So $M \notin D$ and similarly $N \notin E$. Let $P \in Max(A)$, then as $(1+ra)(1+sb) = 0$, either $1+ra \in P$ or $1+sb \in P$. Hence, $P \in D$ or $P \in E$ and so $Max(A) = D \cup E$ and thus $Max(A)$ is Hausdorff. □

Remark 4.6. In the above proposition we have the following:

- (4) $\not\Longrightarrow$ (3) Let A be a domain with exactly two maximal ideals M_1 and M_2. Then, $Max(A)$ is finite, hence discrete, therefore Hausdorff. But as A is not local, by Corollary 4.4, A is not Gelfand.
- (3) $\not\Longrightarrow$ (1) Consider $\mathbb{Z}_{(2)}$. Then $\mathbb{Z}_{(2)}$ is a local domain, hence Gelfand by Corollary 4.4, but it does not have bounded inversion since $2/1 \geq 1/1$ and $2/1$ is not a unit in $\mathbb{Z}_{(2)}$.

Remark 4.7. In **bal**, the proposition above is much stronger. In fact, it is shown in [4], that those four conditions are equivalent in **bal**. In particular, every object in **bal** having bounded inversion is also Gelfand and vice versa. In **baf**, every object with bounded inversion is Gelfand, but the converse is not true as shown in (3) $\not\Longrightarrow$ (1).

Let **bibaf** be the full subcategory of **baf** consisting of objects having the bounded inversion property. Let $S = \{s \in A \mid s \geq 1\}$, then S is multiplicatively closed because $a \in S$ and $b \in S$, give $ab \geq 1$ and so $ab \in S$. For each $A \in$ **baf**, we define $r(A) = S^{-1}A$, where $S^{-1}A$ is the localization of A with respect to S. For $\alpha : A \longrightarrow B$ a **baf**-morphism, define $r(\alpha) : S^{-1}A \longrightarrow S^{-1}B$ by $r(\alpha)(a/s) = \alpha(a)\alpha(s)^{-1}$. Imitating the proof of [4, Proposition 4.2], we get that r is indeed a functor.

Remark 4.8. Notice that S consists of non-zero divisors of A. In fact, any $A \in$ **baf** is torsion free.

Proposition 4.9. ***bibaf*** *is a reflective subcategory of* ***baf****.*

Proof. As mentioned above, we let $r :$ **baf** $\longrightarrow$ **bibaf** be defined by $r(A) = S^{-1}A$, where $S = \{s \in A \mid s \geq 1\}$. To show that r is well-defined, we need to show that $r(A) \in$ **bibaf**. In the case where A is the zero ring, $S^{-1}A = 0$ and so $S^{-1}A \in$ **bibaf**. Now suppose $A \neq 0$. It is well known that the ordering on A extends to an ordering on $S^{-1}A$ via $a/s \geq 0$ if and only if $a \geq 0$, making $S^{-1}A$ an ℓ-ring. If $a/s \in S^{-1}A$, then as A is bounded, $1 \leq |s|$ and $|a| \leq n$ and $|s| \geq 1$ for some $n \in \mathbb{N}$. So $|a/s| = |a|/|s| \leq n$, so $S^{-1}A$ is bounded. Now suppose $na/s \leq b/t$ for all $n \in \mathbb{N}$ where $b/t \geq 0 \in S^{-1}A$. Then $nat \leq bs$ for all $n \in \mathbb{N}$ implying $at \leq 0$ as A is Archimedean. But as A is torsion free, $a \leq 0$ as $t > 0$, so

$S^{-1}A \in \mathbf{baf}$. Moreover, if $a/s \geq 1$, then $s \leq a$ giving $a \in S$ and so a/s is a unit in $S^{-1}A$ as $(a/s)(s/a) = 1$. Thus $S^{-1}A \in \mathbf{bibaf}$. We continue by showing r is a reflector. Let $A \in \mathbf{baf}$ and let $i : A \longrightarrow S^{-1}A$ be defined by $i(a) = a/1$. Suppose $\alpha : A \longrightarrow B$ is a morphism in $\mathbf{baf}$ with $B \in \mathbf{bibaf}$. For each $s \in S$, as $1 \leq s$ we have $1 \leq \alpha(s)$. Therefore $\alpha(s)$ is a unit in B. So let $\beta : S^{-1}A \longrightarrow B$ be defined by $\beta(a/s) = \alpha(a)\alpha(s)^{-1}$ for all $a \in A$ and $s \in S$. Notice that $\big(\beta \circ i\big)(a) = \beta(a/1) = \alpha(a)$, i.e $\beta \circ i = \alpha$ and so the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"'] & S^{-1}A \arrow[dl, dotted, "\beta"] \\ B & \end{tikzcd}$$

Establishing uniqueness of β finishes the proof. For that, notice first that:

$$\begin{aligned} \ker(i) &= \{a \in A \mid a/1 = 0\} \\ &= \{a \in A \mid sa = 0\} \\ &= \{0\}, \end{aligned}$$

where the third equality holds because A is torsion free. So i is injective. On the other hand suppose we have the following diagram

$$A \xrightarrow{\quad i \quad} S^{-1}A \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} B$$

where α_1 and α_2 are $\mathbf{baf}$-morphisms such that $\alpha_1 \circ i = \alpha_2 \circ i$. Then

$$\begin{aligned} \alpha_1(a/s) &= \alpha_1((a/1)(1/s)^{-1}) \\ &= \alpha_1(i(a))\alpha_1(i(s))^{-1} \\ &= \alpha_2(i(a))\alpha_2(i(s))^{-1} \\ &= \alpha_2(a/s) \end{aligned}$$

for all $a/s \in S^{-1}A$. This in fact shows that i is epic as the restriction $B \in \mathbf{bibaf}$ was not used in showing $\alpha_1(a/s) = \alpha_2(a/s)$. As a result of this, we get that if there exists a $\mathbf{baf}$-morphism β' such that $\beta \circ i = \beta' \circ i$ then $\beta = \beta'$, i.e β is unique. Thus $\mathbf{bibaf}$ is a reflective subcategory of $\mathbf{baf}$. □

Corollary 4.10. *$\mathbb{Q}$ is the initial object of **bibaf**.*

Proof. Let $A \in \mathbf{bibaf}$. For all $n \in \mathbb{N}$, n is invertible in A. Thus $\mathbb{Q}$ is isomorphic to a subring of A. Another way of seeing this is by using the fact that reflectors preserve colimits and $S^{-1}\mathbb{Z} = \mathbb{Q}$, when $S = \{n \in \mathbb{Z} : n \geq 1\}$. □

Corollary 4.11. *For $A \in$ **baf** we have $X_A \cong X_{S^{-1}A}$.*

Proof. By the proof of Proposition 4.9, $i : A \longrightarrow S^{-1}A$ is a bimorphism. So by Proposition 2.8 $X_A \cong X_{S^{-1}A}$ via the homeomorphism i^*. □

Theorem 4.12. ***bibal** is a reflective subcategory of **bibaf**.*

Proof. Let $A \in$ **bibaf** and let $r : A \longrightarrow S^{-1}\big(\widetilde{A}\big)$ be defined by $r(a) = f_a/1$. Let $\alpha : A \longrightarrow B$ be a **baf**-morphism where $B \in$ **bibal**. We have the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "s"] \arrow[d, "\alpha"'] & & \widetilde{A} \arrow[d, "t"] \arrow[dll, dotted, "\beta"'] \\
B & & S^{-1}\big(\widetilde{A}\big), \arrow[ll, dotted, "\gamma"]
\end{tikzcd}$$

where $s(a) = f_a$ and $t(x) = x/1$. Notice that the upper diagram commutes as **bal** is a reflective subcategory of **baf**. In fact β is the unique **bal**-morphism that makes the upper triangle commutative. Moreover, the lower triangle commutes as **bibal** is a reflective subcategory of **bal** by [4, Proposition 4.2]. In fact, γ is the unique **bal**-morphism that makes the lower triangle commutes. Thus

$$(\gamma \circ\ r)(a) = \gamma(r(a)) = \gamma(f_a/1) = \gamma(t(f_a)) = \beta(f_a) = \beta(s(a)) = \alpha(a).$$

As γ is a **bal**-morphism, we get the desired conclusion if we show uniqueness. So, suppose $\lambda : S^{-1}\big(\widetilde{A}\big) \longrightarrow B$ is another **bal**-morphism such that $\lambda \circ r = \alpha$. Then $\lambda \circ t \circ s = \alpha$, but β is the unique morphism that satisfies $\beta \circ s = \alpha$, which means $\lambda \circ t = \beta$. Again, γ is the unique morphism satisfying $\gamma \circ t = \beta$ and so $\gamma = \lambda$ and hence uniqueness. $\square$

Now we produce another reflective subcategory of **baf** motivated again from [4], whose objects satisfy the condition that each positive element is a square. Let **scbaf** (square closed) be the full subcategory of **baf** whose objects C satisfy what we'll call the square closed condition:

$$\{a \in C : a \geq 0\} = \{b^2 : b \in C\}.$$

Let $A \in$ **baf** and set $sc(A)$ to be the intersection of all ℓ-subrings of $C(X_A, \mathbb{R})$ satisfying the square closed condition and containing $\varphi_A(A)$. As the square root of every positive continuous function from X_A to $\mathbb{R}$ is again continuous, $C(X_A, \mathbb{R}) \in$ **scbaf**, and so it is easy to see that $sc(A) \in$ **scbaf**. In fact, we can give a concrete construction of $sc(A)$ by imitating [4, Proposition 4.6] with very little modification. For each ℓ-subring B of $C(X_A, \mathbb{R})$, let B' be the ℓ-subring of $C(X_A, \mathbb{R})$ generated by $B \cup \{\sqrt{b} : b \in B^+\}$. As B' is an ℓ-subring of $C(X_A, \mathbb{R})$, B' is an object in **baf** such that $B \subseteq B' \subseteq C(X_A, \mathbb{R})$. Moreover, by construction, each $b \in B^+$ has a unique positive square root in B'.

Lemma 4.13. *$sc(A) = \bigcup_{n=0}^{\infty} A_n$, where $A_0 = \varphi_A(A)$ and $A_{n+1} = A_n'$.*

Lemma 4.14. *sc : **baf** $\longrightarrow$ **scbaf** is a functor.*

Proposition 4.15. ***scbaf** is a reflective subcategory of **baf**.*

Proof. Let i_A be the morphism $\varphi_A : A \longrightarrow C(X_A, \mathbb{R})$ viewed as a map into $sc(A)$. This is possible as we have seen in Lemma 4.13 that $\varphi_A(A) \subseteq sc(A)$. Suppose $C \in$ **scbaf** and $\alpha : A \longrightarrow C$ is a **baf**-morphism. By Lemma 4.14, $sc(\alpha) : sc(A) \longrightarrow sc(C)$ is a **baf**-morphism. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[d, "\alpha"'] & sc(A) \arrow[d, "sc(\alpha)"] \\
C \arrow[r, "i_C"'] & sc(C)
\end{tikzcd}$$

As C is square closed, we have $sc(C) = \varphi_C(C)$. Set $\beta = i_C^{-1} \circ sc(\alpha)$. Then $\beta : sc(A) \longrightarrow C$ is a **baf**-morphism. But by commutativity of the diagram above we have

$$\beta \circ i_A = i_C^{-1} \circ sc(\alpha) \circ i_A = i_C^{-1} \circ i_C \circ \alpha = \alpha.$$

So we found β that makes the following diagram commute

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i_A"'] & C \\ sc(A) \arrow[ur, dotted, "\beta"'] & \end{tikzcd}$$

Suppose there exists $\gamma : sc(A) \longrightarrow C$ that also satisfies $\gamma \circ i_A = \alpha$, then $\gamma \circ i_A = \beta \circ i_A$ implying $\gamma|_{\varphi_A(A)} = \beta|_{\varphi_A(A)}$, therefore $\gamma|_{A_0} = \beta|_{A_0}$. Suppose $0 \leq a \in A$, then $\sqrt{a} \in sc(A)$ and $\gamma(\sqrt{a}) = \beta(\sqrt{a})$. But this means $\big(\gamma(\sqrt{a})\big)^2 = \big(\beta(\sqrt{a})\big)^2 = \alpha(a)$. Thus α and β agree on A_1. An induction argument gives $\alpha = \beta$ and hence β is unique. □

Theorem 4.16. ***scbal*** *is a reflective subcategory of* ***scbaf***

Proof. Let $A \in$ **scbaf** and let $r : A \longrightarrow sc\big(\widetilde{A}\big)$ be defined by $r(a) = i_{\widetilde{A}}(f_a)$, where $i_{\widetilde{A}}$ is the morphism $\varphi_{\widetilde{A}} : \widetilde{A} \longrightarrow C(X_{\widetilde{A}}, \mathbb{R})$ viewed into $sc(\widetilde{A})$. Let $\alpha : A \longrightarrow B$ be a **baf**-morphism where $B \in$ **scbal**. We have the following diagram:

$$\begin{tikzcd} A \arrow[rr, "s"] \arrow[d, "\alpha"'] & & \widetilde{A} \arrow[d, "i_{\widetilde{A}}"] \arrow[dll, dotted, "\beta"'] \\ B & & sc\big(\widetilde{A}\big) \arrow[ll, dotted, "\gamma"] \end{tikzcd}$$

Where $s(a) = f_a$. Notice that the upper diagram commutes as **bal** is a reflective subcategory of **baf**. In fact, β is the unique **bal**-morphism that makes the upper triangle commute. Moreover, the lower triangle commutes as **scbal** is a reflective subcategory of **bal** by [4, Proposition 4.6]. In fact, γ is the unique **bal**-morphism that makes the lower triangle commute. Thus :

$$\gamma \circ\ r(a) = \gamma(r(a)) = \gamma\big(i_{\widetilde{A}}(f_a)\big) = \beta(f_a) = \beta(s(a)) = \alpha(a).$$

As γ is a **bal**-morphism, we get the desired conclusion if we show γ is unique, but that can be established in a routine way as in Theorem 4.12. □

Example 4.17. Consider the ring of integers $\mathbb{Z}$ and let $A = sc(\mathbb{Z})$. By the construction above and because the notions subring and ℓ-subring coincide as $C(X_{\mathbb{Z}}, \mathbb{R}) \cong \mathbb{R}$ is totally ordered, we see that $sc(\mathbb{Z})$ is a subring of $\mathbb{R}$ integral over $\mathbb{Z}$, in particular $\sqrt{2} \in sc(\mathbb{Z})$. Therefore A is not an $\mathbb{R}$-algebra. So $A \in$ **scbaf** $-$ **scbal**.

Remark 4.18. Example 4.17 shows that **scbaf** and **bibaf** are not comparable. $A = sc(\mathbb{Z}) \in$ **scbaf** but as $sc(\mathbb{Z}) \cap \mathbb{Q}$ is also integral over $\mathbb{Z}$, we have $sc(\mathbb{Z}) \cap \mathbb{Q} = \mathbb{Z}$ which implies that $sc(\mathbb{Z})$ does not have bounded inversion as $\sqrt{2} > 1$ and $\sqrt{2}$ is not invertible in A. Conversely, $\mathbb{Q}$ is not square closed but does have bounded inversion.

As we have noticed in Proposition 4.5, not every Gelfand object in **baf** has bounded inversion. Let **gbaf** denote the full subcategory of **baf** consisting of Gelfand rings.

Theorem 4.19. ***gbaf*** *is not a reflective subcategory of* ***baf***.

Proof. Let us suppose that **gbaf** is a reflective subcategory of **baf** and let g be the reflector. Then we can consider the following diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "g"] \arrow[d, "i"'] & g(\mathbb{Z}) \arrow[dl, dotted, "\phi"] \\ \mathbb{R} & \end{tikzcd}$$

Notice that as our ℓ-ring homomorphisms are unital and going into $\mathbb{R}$, we get that the diagram commute and that ϕ is unique. The uniqueness of ϕ will show that $g(\mathbb{Z})$ has only one maximal ℓ-ideal. For if $M_1 \neq M_2 \in X_{g(\mathbb{Z})}$, then we get two morphisms $\beta_1 : g(\mathbb{Z}) \longrightarrow \mathbb{R}$ and $\beta_2 : g(\mathbb{Z}) \longrightarrow \mathbb{R}$ with kernels M_1 and M_2 respectively, hence β_1 and β_2 are distinct, having distinct kernels, and they make the diagram above commute which is a contradiction. Therefore $X_{g(\mathbb{Z})} = 0$ which means $g(\mathbb{Z})$ is isomorphic to a subring of $\mathbb{R}$ and so $g(\mathbb{Z})$ is a totally ordered Archimedean integral domain. Now consider the following diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "g"] \arrow[d, "i"'] & g(\mathbb{Z}) \arrow[dl, dotted, "\phi"] \\ \mathbb{Z}_{(2)} & \end{tikzcd}$$

As $-2+3 = 1$ and $g(\mathbb{Z})$ is Gelfand, there exist r and s in $g(\mathbb{Z})$ such that $(1-2r)(1+3s) = 0$. Therefore, 2 is a unit or 3 is a unit in $g(\mathbb{Z})$. If 2 is a unit, then 2 is also a unit in $\mathbb{Z}_{(2)}$, but it isn't. So 3 is a unit. Repeat the same argument with $\mathbb{Z}_{(3)}$ instead of $\mathbb{Z}_{(2)}$ to see that 3 cannot be a unit. Thus there is no reflector from **baf** to **gbaf**. $\square$

Now we take a look at one more full subcategory of **baf** denoted **ubaf**, consisting of those $A \in$ **baf** which are uniformly complete.

Let $A \in$ **baf**. Then the metric space completion of A in $C(X_A, \mathbb{R})$, denoted by $\overline{A}$ instead of $\overline{\varphi_A(A)}$ to ease notation, is in **baf**. To see that, let $x, y \in \overline{A}$, then there exist a_n and b_n in A such that $a_n \to x$ and $b_n \to y$. By continuity of the operations on A, $a_n + b_n \to x + y$, $a_n \vee b_n \to x \vee y$ and $a_n b_n \to xy$ (similarly for the meet operation). By completeness of $\overline{A}$, $x + y$, $x \vee y$ and xy are all in $\overline{A}$. Thus, $\overline{A}$ is a sub-**baf** of $C(X_A, \mathbb{R})$.

Proposition 4.20. ***ubaf*** *is a reflective subcategory of* ***baf****.*

Proof. Let $\alpha : A \longrightarrow C$ be a **baf**-morphism where $C \in$ **ubaf**. There exists a unique **bal**-morphism $\beta : C(X_A, \mathbb{R}) \longrightarrow C(X_C, \mathbb{R})$. Consider its restriction to $\overline{A}$. We get the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d, "\alpha"'] & \overline{A} \arrow[d, "\beta"] \\ C \arrow[r, "\varphi_C"'] & C(X_C, \mathbb{R}) \end{tikzcd}$$

If we show that $\beta[\overline{A}] \subseteq \varphi_C(C)$, then we would get that $\beta \circ \alpha = \varphi_A$ and we are done except for uniqueness. Suppose $x \in \overline{A}$, then $x = \lim_{n\to\infty} \varphi_A(x_n)$ for some $(x_n) \in A$. As $\varphi_A(x_n)$ converges in $\overline{A}$, (x_n) is Cauchy in A. As α is uniformly continuous, $\alpha((x_n))$ is also Cauchy in C (this will be shown later in 4.25) and as C is complete, $\lim_{n\to\infty} \alpha(x_n) \in C$. So we have

$$\beta(x) = \beta(\lim_{n\to\infty} \varphi_A(x_n) = \lim_{n\to\infty} \beta(\varphi_A(x_n)) = \lim_{n\to\infty} \varphi_C \alpha(x_n) = \varphi_C(\lim_{n\to\infty} \alpha(x_n)) = \varphi_C(c),$$

for some $c \in C$. Note that by the universal mapping property of metric space completion, we get a unique continuous map $f : \overline{A} \longrightarrow C$ such that $f \circ \varphi_A = \alpha$. Since $\varphi_C \circ \beta : \overline{A} \longrightarrow C$ satisfies $(\varphi_C \circ \beta) \circ \varphi_A = \alpha$, uniqueness of β follows. In fact, $f = \varphi_C \circ \beta$. □

Corollary 4.21. ***ubal*** *is a reflective subcategory of* ***ubaf****.*

Proof. The proof is similar to the proofs of Theorems 4.12 and 4.16. □

Proposition 4.22. *For X compact topological space, $C(X, \mathbb{Z})$ is uniformly complete.*

Proof. The proof of this is standard. □

Corollary 4.23. ***ubal*** *is a proper subcategory of* ***ubaf*** *and it is incomparable with* ***bibaf****.*

Proof. $\mathbb{Z}$ is uniformly complete but does not have bounded inversion. Conversely, in [4, Proposition 4.2], the authors show that $PC(X, \mathbb{R})$ has bounded inversion and is not uniformly complete. □

Theorem 4.24. *In general, $A \in$* ***ubaf*** *does not imply $\widetilde{A} \in$* ***ubaf****.*

Proof. Let $A = C(X, \mathbb{Z})$ where X is an infinite Stone space (for example the Cantor set). Then $A = FC(X, \mathbb{Z})$, the set of all continuous finitely valued functions into $\mathbb{Z}$ on X, [3, Proposition 5.4]. Moreover, if $f \in FC(X, \mathbb{R})$, say $f[X] = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let $C_i = f^{-1}(\alpha_i)$, then $f = \sum_{i=1}^{n} \alpha_i \chi_{C_i}$, χ_{C_i} is the characteristic function of the set C_i. As $\chi_{C_i} \in FC(X, \mathbb{Z})$, we get that $FC(X, \mathbb{R})$ is generated as an $\mathbb{R}$-vector space by $FC(X, \mathbb{Z})$ giving $\widetilde{C(X, \mathbb{Z})} \cong FC(X, \mathbb{R})$. Note that as X is a Stone space, If $x \neq y$, there exists a clopen U with $x \in U$ and $y \notin U$ and so $\chi_U \in FC(X, \mathbb{R})$ separates x and y. So the inclusion map $i : FC(X, \mathbb{R}) \longrightarrow C(X, \mathbb{R})$ separates points. Thus by the Stone-Weierstrass theorem, $FC(X, \mathbb{R})$ is dense in $C(X, \mathbb{R})$. It is clear that if X is the Cantor set then $FC(X, \mathbb{R})$ is a proper sub-algebra of $C(X, \mathbb{R})$, which means that $FC(X, \mathbb{R})$ cannot be complete and so $\widetilde{C(X, \mathbb{Z})}$ is not complete; however, as mentioned before, $C(X, \mathbb{Z})$ is complete. □

Theorem 4.25. *If $A \in$* ***ubaf*** *and $M \in X_A$, then A/M is in* ***ubaf****.*

Proof. It should be noted that the proof for vector lattices can be found in [8, Theorem 59.3], but we modify the proof so that it fits our discussion, we also give some justifications for some of the steps. To ease notation, let $B = A/M$ and let $\pi : A \longrightarrow B$. Since π is uniformly continuous π send Cauchy sequences to Cauchy sequences. Now we show that if $\pi(a_n)$ is Cauchy in B, then there exists a subsequence $\pi(a_{n_k})$ and a corresponding sequence a'_k in A with $\pi(a'_k) = \pi(a_{n_k})$ for all k such that a'_k is Cauchy in A. Suppose $\pi(a_n)$ is Cauchy in B and let $\pi(a_{n_k})$ be a subsequence with $||\pi(a_{n_{k+1}}) - \pi(a_{n_k})|| \leq \frac{1}{2^k}$ for all k. Hence $||\pi(a_{n_{k+1}} - a_{n_k})|| \leq \frac{1}{2^k}$ and so $|\pi(a_{n_{k+1}} - a_{n_k})| \leq \frac{1}{2^k}$. Let $b_k = a_{n_{k+1}} - a_{n_k}$. By definition of the ordering on B, there exists $b'_k \in A$ with $|b'_k| + M = |b_k| + M$, i.e $\pi(|b'_k|) = \pi(|b_k|)$ and $2^k |b'_k| \leq 1$ for all k. Define $a'_k = a_{n_1} + b'_1 + \cdots + b'_{k-1}$. Then $\pi(a'_k) = \pi(a_{n_k})$ and $||a'_{k+1} - a'_k|| = ||b'_k|| \leq \frac{1}{2^k}$. Therefore (a'_k) is a Cauchy sequence. Now we are ready to show that B is uniformly complete. Suppose $\pi(a_n)$ is Cauchy in B. Then there exists $\pi(a_{n_k})$ and a'_k with $\pi(a'_k) = \pi(a_{n_k})$, where a'_k is Cauchy for each k. But as A is uniformly complete, $a'_k \to a \in A$. As π is continuous, $\pi(a'_k) \to \pi(a)$ and hence $\pi(a_{n_k}) \to \pi(a)$. Thus $\pi(a_{n_k}) \to \pi(a) \in B$ uniformly. Thus $B \in$ **ubaf**. □

Corollary 4.26. *If $A \in$ **ubaf** and $M \in X_A$ then A/M is isomorphic to either $\mathbb{R}$ or $\mathbb{Z}$.*

Proof. By Theorem 4.25, $A/M \in$ **ubaf**. So A/M is isomorphic to a subring of $\mathbb{R}$. But those subrings that are uniformly complete are either $\mathbb{Z}$ or $\mathbb{R}$ as any non-discrete subgroup of $\mathbb{R}$ is dense in $\mathbb{R}$. □

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