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CONSTRUCTING HOM-(TRI)DENDRIFORM FAMILY ALGEBRAS

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Abstract. Hom-(tri)dendriform family algebras and Hom-pre-Lie family algebras are introduced and studied. Moreover we explore the connections between these categories of family Hom-algebras.

1. Introduction

Dendriform algebras are algebras with two operations, which dichotomize the notion of associative algebra. They are introduced by Loday [16] with motivation from algebraic K-theory and are connected to several areas in mathematics and physics, including homology [9], arithmetics [15], operads [17], Hopf algebras [18, 19, 22], combinatorics and quantum field theory [8], [10] where they occur in the theory of renormalization of Connes and Kreimer [3, 4, 5]. Later a tridendriform algebra which is a vector space equipped with 3 binary operations satisfying seven relations were introduced by Loday and Ronco in their study of polytopes and Koszul duality [19].

Rota-Baxter operators have appeared in a wide range of areas in pure and applied mathematics. The paradigmatic example of Rota-Baxter operators concerns the integration by parts formula of continuous functions. The algebraic formulation of Rota-Baxter algebra find their origin in the work of the American mathematician Glen E. Baxter [2] in probability study of fluctuation theory and it is proved that a Rota-Baxter algebra of weight zero (resp. weight $\lambda \neq 0$) possesses a dendriform (resp. tridendriform) algebra structure [1], [6]. Rota-Baxter algebras were intensively studied on the one hand by G.C. Rota in connection with combinatorics and on the other hand by A. Connes and D. Kreimer in connection with mathematical physics in their Hopf algebra approach fof renormalizing of quantum field theory. Therefore, an important development including Rota-Baxter algebras and their connections to other algebraic structure is obtained.

The operations on an algebra may be replaced by a family of operations indexed by a semigroup, giving rise to the same algebraic structure in "family versions".

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The notion of "Rota-Baxter family" appeared in 2007 is the first example of this situation [7] (see also [14]) about algebraic aspects of renormalization in Quantum Field Theory. This terminology was suggested to the authors by Li Guo, who further duscussed the underlying structure under the name Rota-Baxter Family algebra [12] which is a generalization of Rota-Baxter algebras. Since then, various other kinds of family algebraic structures have been defined [11, 23, 24, 25].

The aim of this paper is to extend the notion of (tri)-dendrifom family algebras and pre-Lie family algebras to the Hom-setting. Recall that the theory of Hom-algebras first appeared in the study of quasi-deformations of Lie algebras of vector fields, including q-deformations of Witt algebras and Virasoro algebras with the introduction of Hom-Lie algebras by J.T. Hartwig, D. Larsson, and S.D. Silvestrov [13].

The paper is organized as follows: In section two, we introduce the concepts of Hom-associative Rota-Baxter family algebras and then prove various constructions of them from associative Rota-Baxter algebras. Furtheremore, we give a method to induce Hom-associative Rota-Baxter algebras from Hom-associative Rota-Baxter family algebras. In section three, we introduce the notions of Hom-(tri)dendriform) and Hom-pre-Lie family algebras. Finally, some relations between these family Hom-algebras are given.

Throughout this paper, all vector spaces and algebras are meant over a ground field $\mathbb{K}$ and Ω designates a semigroup whose elements will be denoted by $\lambda, \mu, \nu, \xi, \cdots$.

2. Hom-associative Rota-Baxter family algebras

In this section, we introduce the concepts of Hom-associative Rota-Baxter family algebras and then prove various constructions of them from associative Rota-Baxter algebras. We also give a method to induce Hom-associative Rota-Baxter algebras from Hom-associative Rota-Baxter family algebras.

Definition 2.1. *A* **Rota-Baxter operator of weight** $\theta \in \mathbb{K}$ *on a Hom-algebra* $(A, \cdot, \alpha)$ *is a linear operator* $R : A \to A$ *that satisfies*

$$R \circ \alpha = \alpha \circ R, \tag{2.1}$$

$$R(a) \cdot R(b) = R\big(R(a) \cdot b + a \cdot R(b) + \theta a \cdot b\big), \text{ for all} a, b \in A \tag{2.2}$$

Equation (2.2) is called the Rota-Baxter equation. Then $(A, \cdot, \alpha, R)$ *is called a* **Rota-Baxter Hom-algebra of weight** λ*. If further* $\cdot$ *is commutative (resp. Hom-associative), then* $(A, \cdot, \alpha, R)$ *is called a commutative Rota-Baxter Hom-algebra (resp. a Hom-associative Rota-Baxter algebra* [20]*) of weight* θ*.*

Definition 2.2. *A* **Rota-Baxter family** *of weight* $\theta \in \mathbb{K}$ *on a Hom-algebra* $(A, \cdot, \alpha)$ *is a collection of linear operators* $\{R_\lambda, \lambda \in \Omega\}$ *on* $(A, \cdot, \alpha)$ *such that*

$$\alpha \circ R_\lambda = R_\lambda \circ \alpha \text{ for all } \lambda \in \Omega, \tag{2.3}$$

$$R_\lambda(a) \cdot R_\eta(b) = R_{\lambda\eta}\Big(R_\lambda(a) \cdot b + a \cdot R_\eta(b) + \theta a \cdot b\Big), \tag{2.4}$$

$$\text{for all } \; a, b \in A \; \text{ and } \; \lambda, \eta \in \Omega.$$

Then the quadruple $(A, \cdot, \alpha, \{R_\lambda, \lambda \in \Omega\})$ *is called a* **Rota-Baxter family Hom-algebra** *of weight* θ*. If further* $\cdot$ *is commutative, then* $(A, \cdot, \alpha, \{R_\lambda, \lambda \in \Omega\})$ *is called a commutative Rota-Baxter family Hom-algebra of weight* θ*.*

Definition 2.3. *Let $(A, \cdot, \alpha, \{R_\lambda, \lambda \in \Omega\})$ and $(B, \intercal, \beta, \{P_\eta, \eta \in \Omega\})$ be two Rota-Baxter family Hom-algebras of weight θ. A map $f : A \to B$ is called* **a Rota-Baxter family Hom-algebra morphism** *if f is Hom-algebras morphism and $f \circ R_\lambda = R_\lambda \circ f$ for each $\lambda \in \Omega$.*

Now, we give a link between Rota-Baxter family Hom-algebras and ordinary Rota-Baxter Hom-algebra as follows.

Let $(A, \cdot, \alpha)$ be a Hom-algebra. Consider the space $A \otimes \mathbb{K}\Omega$ with the linear map

$$\tilde{\alpha} : A \otimes \mathbb{K}\Omega \to A \otimes \mathbb{K}\Omega,\ \tilde{\alpha}(a \otimes \lambda) = \alpha(a) \otimes \lambda.$$

Moreover, the multiplication on A induces a multiplication $\bullet$ on $A \otimes \mathbb{K}\Omega$ by

$$(a \otimes \lambda) \bullet (b \otimes \eta) := a \cdot b \otimes \lambda\eta.$$

It is easy to see that $(A \otimes \mathbb{K}\Omega, \bullet, \tilde{\alpha})$ is a Hom-algebra. Moreover, if $(A, \cdot, \alpha, \{R_\lambda, \lambda \in \Omega\})$ is a Rota-Baxter family Hom-algebra of weight θ then, $(A \otimes \mathbb{K}\Omega, \bullet, \tilde{\alpha}, R\})$ is a Rota-Baxter Hom-algebra of weight θ, where $R : A \otimes \mathbb{K}\Omega \to A \otimes \mathbb{K}\Omega, x \otimes \lambda \mapsto R_\lambda(x) \otimes \lambda$.

Definition 2.4. *Let $\mathcal{A} := (A, \alpha)$ be a Hom-associative algebra. A Rota-Baxter family of weight $\theta \in \mathbb{K}$ on $\mathcal{A}$ is a collection $\{R_\lambda, \lambda \in \Omega\}$ of linear maps on A satisfying (2.3) and (2.4). Then the quadruple $(A, \cdot, \alpha, \{R_\lambda, \lambda \in \Omega\})$ is called a* **Hom-associative Rota-Baxter family algebra** *of weight θ.*

If $\alpha = Id_A$, we get the notion of a Rota-Baxter family of weight $\theta \in \mathbb{K}$ on the associative algebra $(A, \cdot)$.

Proposition 2.5. *Let α be a morphism of a Rota-Baxter family $\{R_\lambda, \lambda \in \Omega\}$ of weight θ on an associative algebra $(A, \cdot)$. Then $\{R_\lambda, \lambda \in \Omega\}$ is a Rota-Baxter family of weight θ on the induced Hom-associative algebra $\mathcal{A}_\alpha := (A, \cdot_\alpha, \alpha)$.*

Proof. First, pick $\lambda \in \Omega$ and $a, b \in A$ then, we have $\alpha \circ R_\lambda = R_\lambda \circ \alpha$ and $\alpha(a \cdot b) = \alpha(a) \cdot \alpha(b)$ since α is a morphism and hence, $\alpha(a \cdot_\alpha b) = a \cdot_\alpha b$ (multiplicativity). Next, we get by (2.4) for all $\lambda, \eta \in \Omega$ and $a, b \in A$,

$$\begin{aligned} R_\lambda(a) \cdot_\alpha R_\eta(b) &= \alpha\Big(R_\lambda(a) \cdot R_\eta(b)\Big) = \alpha\Big(R_{\lambda\eta}\Big(R_\lambda(a) \cdot b + a \cdot R_\eta(b) + \theta a \cdot b\Big)\Big) \\ &= R_{\lambda\eta}\Big(R_\lambda(a) \cdot_\alpha b + a \cdot_\alpha R_\eta(b) + \theta a \cdot_\alpha b\Big). \end{aligned}$$

□

Recall that given an associative algebra $(A, \cdot)$, an self-morphism $\alpha \in End(A)$ is said to be an element of the centroid if $\alpha(x \cdot y) = \alpha(x) \cdot y = x \cdot \alpha(y)$ for all $x, y \in A$. The centroid of A is defined by:

$$Cent(A) := \{\alpha \in End(A), \alpha(x \cdot y) = \alpha(x) \cdot y = x \cdot \alpha(y),\ \forall x, y \in A\}.$$

Now, we also have:

Proposition 2.6. *Let $\{R_\lambda, \lambda \in \Omega\}$ be a Rota-Baxter family of weight θ on an associative algebra (A, μ) and $\alpha \in Cent(A)$ commuting with R_λ for all $\lambda \in \Omega$. Let set for all $x, y \in A$:*

$$\mu^1_\alpha(x, y) = \mu(\alpha(x), y) \text{ and } \mu^2_\alpha(x, y) = \mu(x, \alpha(y)).$$

Then $\mathcal{A}_1 : (A, \mu^1_\alpha, \alpha)$ and $\mathcal{A}_2 : (A, \mu^2_\alpha, \alpha)$ are Hom-associative algebras and $\{R_\lambda, \lambda \in \Omega\}$ is a Rota-Baxter family of weight θ of both of them.

Proof. It is clear that $\mathcal{A}_1 : (A, \mu^1_\alpha, \alpha)$ and $\mathcal{A}_2 : (A, \mu^2_\alpha, \alpha)$ are Hom-associative algebras. Pick $x, y \in A$, $\lambda, \eta \in \Omega$ and set $\mu(x, y) = x \cdot y$, then:

$$\begin{aligned}
&\mu^1_\alpha(R_\lambda(x), R_\eta(y)) = \mu(\alpha(R_\lambda(x)), R_\eta(y)) = \alpha(\mu(R_\lambda(x), R_\eta(y)))\\
&= \alpha\Big(R_{\lambda\eta}\Big(R_\lambda(x) \cdot y + x \cdot R_\eta(y) + \theta x \cdot y\Big)\Big) \text{ (by (2.4))}\\
&= R_{\lambda\eta}\Big(\alpha(R_\lambda(x)) \cdot y + \alpha(x) \cdot R_\eta(y) + \theta\alpha(x) \cdot y\Big)\\
&= R_{\lambda\eta}\Big(\mu^1_\alpha(R_\lambda(x), y) + \mu^1_\alpha(x, R_\eta(y)) + \theta\mu^1_\alpha(x, y)\Big).
\end{aligned}$$

Similarly, we get also

$$\mu^2_\alpha(R_\lambda(x), R_\eta(y)) = R_{\lambda\eta}\Big(\mu^2_\alpha(R_\lambda(x), y) + \mu^2_\alpha(x, R_\eta(y)) + \theta\mu^2_\alpha(x, y)\Big)$$

□

Let $(A, \cdot, \alpha)$ be a Hom-associative algebra. Consider the space $A \otimes \mathbb{K}\Omega$ with the linear map

$$\tilde{\alpha} : A \otimes \mathbb{K}\Omega \to A \otimes \mathbb{K}\Omega,\ \tilde{\alpha}(a \otimes \lambda) = \alpha(a) \otimes \lambda.$$

Moreover, the multiplication on A induces a multiplication $\bullet$ on $A \otimes \mathbb{K}\Omega$ by

$$(a \otimes \lambda) \bullet (b \otimes \eta) := a \cdot b \otimes \lambda\eta.$$

It is easy to see that $(A \otimes \mathbb{K}\Omega, \bullet, \tilde{\alpha})$ is a Hom-associative algebra.

Theorem 2.7. *Let $\{R_\lambda, \lambda \in \Omega\}$ be a Rota-Baxter family of weight θ on the Hom-associative algebra $(A, \cdot, \alpha)$. Then the map*

$$R : A \otimes \mathbb{K}\Omega \to A \otimes \mathbb{K}\Omega,\ R(a \otimes \lambda) := R_\lambda(a) \otimes \lambda$$

is a Rota-Baxter operator of weight θ on the Hom-associative algebra $(A \otimes \mathbb{K}\Omega, \bullet, \tilde{\alpha})$.

Proof. First, it is clear that $\tilde{\alpha} \circ R = R \circ \tilde{\alpha}$. Next, pick $a, b \in A$ and $\lambda, \eta \in \Omega$ then, we have:

$$\begin{aligned}
& R(a \otimes \lambda) \bullet R(b \otimes \eta)\\
= \ & \Big(R_\lambda(x) \otimes \lambda\Big) \bullet \Big(R_\eta(y) \otimes \eta\Big)\\
= \ & \Big(R_\lambda(a) \cdot R_\eta(b)\Big) \otimes \lambda\eta\\
= \ & R_{\lambda\eta}\Big(R_\lambda(a) \cdot b + a \cdot R_\eta(b) + \theta x \cdot y\Big) \otimes \lambda\eta\\
= \ & R\Big(\Big(R_\lambda(a) \cdot b + a \cdot R_\eta(b) + \theta a \cdot b\Big) \otimes \lambda\eta\Big)\\
= \ & R\Big(R_\lambda(a) \cdot b \otimes \lambda\eta + a \cdot R_\eta(b) \otimes \lambda\eta + \theta a \cdot b \otimes \lambda\eta\Big)\\
= \ & R\Big(\Big(R_\lambda(a) \otimes \lambda\Big)(b \otimes \eta) + (a \otimes \lambda)\Big(R_\eta(b) \otimes \eta\Big) + \theta(a \otimes \lambda) \bullet (b \otimes \eta)\Big)\\
= \ & R\Big(R(a \otimes \lambda) \bullet (b \otimes \eta) + (a \otimes \lambda) \bullet R(b \otimes \eta) + \theta(a \otimes \lambda) \bullet (b \otimes \eta)\Big).
\end{aligned}$$

□

3. Hom-(tri)dendriform family algebras

In this section, we recall the concepts of Hom-(tri)dendriform algebras [20] and Hom-pre-Lie algebras [21] and then introduce Hom-(tri)dendriform and Hom-pre-Lie family algebras. We then give on the one hand a method to induce Hom-(tri)dendriform) algebras from Hom-(tri)dendrifom family algebras and on the other hand a method to obtain Hom-pre-Lie algebras from Hom-pre-Lie family algebras. Furtheremore, some relations between these family Hom-algebras are given.

Definition 3.1. [20] *A* **Hom-dendriform algebra** *is a is a quadruple* $(D, \prec, \succ, \alpha)$ *consisting of a vector space* D *on which the operations* $\prec, \succ \colon D \otimes D \to D$ *and* $\alpha : D \to D$ *are linear maps satisfying*

$$\alpha \circ \prec = \prec \circ \alpha^{\otimes 2} \text{ and } \alpha \circ \succ = \succ \circ \alpha^{\otimes 2} \tag{3.1}$$
$$(x \prec y) \prec \alpha(z) = \alpha(x) \prec (y \prec z + y \succ z), \tag{3.2}$$
$$(x \succ y) \prec \alpha(z) = \alpha(x) \succ (y \prec z), \tag{3.3}$$
$$(x \prec y + x \succ y) \succ \alpha(z) = \alpha(x) \succ (y \succ z). \tag{3.4}$$

for all x, y, z in D.

Definition 3.2. [20] *A* **Hom-tridendriform algebra** *is a is a quintuple* $(T, \prec, \succ, \cdot, \alpha)$ *consisting of a vector space* D *on which the operations* $\prec, \succ, \cdot : D \otimes D \to D$ *and* $\alpha : D \to D$ *are linear maps satisfying*

$$\alpha \circ \prec = \prec \circ \alpha^{\otimes 2},\ \alpha \circ \succ = \succ \circ \alpha^{\otimes 2} \text{ and } \alpha \circ \cdot = \cdot \circ \alpha^{\otimes 2}, \tag{3.5}$$
$$(x \prec y) \prec \alpha(z) = \alpha(x) \prec (y \prec z + y \succ z + y \cdot z), \tag{3.6}$$
$$(x \succ y) \prec \alpha(z) = \alpha(x) \succ (y \prec z), \tag{3.7}$$
$$(x \prec y + x \succ y + x \cdot y) \succ \alpha(z) = \alpha(x) \succ (y \succ z), \tag{3.8}$$
$$(x \succ y) \cdot \alpha(z) = \alpha(x) \succ (y \cdot z), \tag{3.9}$$
$$(x \prec y) \cdot \alpha(z) = \alpha(x) \cdot (y \succ z), \tag{3.10}$$
$$(x \cdot y) \prec \alpha(z) = \alpha(x) \cdot (y \prec z), \tag{3.11}$$
$$(x \cdot y) \cdot \alpha(z) = \alpha(x) \cdot (y \cdot z). \tag{3.12}$$

Remark. *Any Hom-tridendriform algebra gives a Hom-dendriform algebra by setting* $x \cdot y = 0$ *for all* $x, y \in T$*. Furtheremore, a Hom-(tri)dendriform algebra reduces to a (tri)dendriform algebra when* $\alpha = Id$*.*

Next, we gereneralize in Hom-case, the concept of dendriform family algebras [23] which was proposed as a generalization of dendriform algebras invented by Loday in the study of algebraic K-theory [16].

Definition 3.3. *A* **Hom-dendriform family algebra** *is a* $\mathbb{K}$*-Hom-module* (D, α) *with a family of binary operations* $\{\prec_\lambda, \succ_\lambda, \lambda \in \Omega\}$ *such that for* $x, y, z \in A$ *and* $\lambda, \eta \in \Omega$*,*

$$\alpha \circ \prec_\lambda = \prec_\lambda \circ \alpha^{\otimes 2} \text{ and } \alpha \circ \succ_\lambda = \succ_\lambda \circ \alpha^{\otimes 2} \tag{3.13}$$
$$(x \prec_\lambda y) \prec_\eta \alpha(z) = \alpha(x) \prec_{\lambda\eta} (y \prec_\eta z + y \succ_\lambda z), \tag{3.14}$$
$$(x \succ_\lambda y) \prec_\eta \alpha(z) = \alpha(x) \succ_\lambda (y \prec_\eta z), \tag{3.15}$$
$$(x \prec_\eta y + x \succ_\lambda y) \succ_{\lambda\eta} \alpha(z) = \alpha(x) \succ_\lambda (y \succ_\eta z). \tag{3.16}$$

Remark. *When the semigroup Ω is taken to be the trivial monoid with one single element, a Hom-dendriform family algebra is precisely a Hom-dendriform algebra. Furtheremore, a Hom-dendriform family algebra reduces to a dendriform family algebra when $\alpha = Id$.*

We can prove easily the following result which gives a method for constructing Hom-dendriform family algebras from dendriform family algebras.

Theorem 3.4. *Let $(A, \{\prec_\lambda, \succ_\lambda, \lambda \in \Omega\})$ be a dendriform family algebra and $\alpha : A \to A$ be a self-morphism of dendriform family algebra. Then $(A, \{\prec_\lambda^\alpha, \succ_\lambda^\alpha, \lambda \in \Omega\}, \alpha)$ is Hom-dendriform family algebra where*

$$x \prec_\lambda^\alpha y := \alpha(x \prec_\lambda y) \text{ and } x \succ_\lambda^\alpha y := \alpha(x \succ_\lambda y) \text{ for all } \lambda \in \Omega \text{ and } x, y \in A.$$

Proof. Straightforward computations. □

Proposition 3.5. *A Rota-Baxter family $\{R_\lambda, \lambda \in \Omega\}$ of weight $\theta \in \mathbb{K}$ on a Hom-associative algebra $(D, \cdot, \alpha)$ induces a Hom-dendriform family algebra $(D, \{\prec_\lambda, \succ_\lambda, \lambda \in \Omega\}, \alpha)$ where*

$$a \prec_\lambda b := a \cdot R_\lambda(b) + \theta a \cdot b \text{ and } a \succ_\lambda b := R_\lambda(a) \cdot (b) \text{ for all } \lambda \in \Omega \text{ and } a, b \in D.$$

Proof. First, the multiplicativity of α with respect to $\cdot$ and Condition (2.3) imply the multiplicativity of α with respect to $\prec_\lambda$ and $\succ_\lambda$ for any $\lambda \in \Omega$. Next, for all $x, y, z \in D$ and $\lambda, \eta \in \Omega$, we compute

$$\begin{aligned}
&(x \prec_\lambda y) \prec_\eta \alpha(z) = (x \cdot R_\lambda(y) + \theta x \cdot y) \prec_\eta \alpha(z) \\
&= (x \cdot R_\lambda(y) + \theta x \cdot y) \cdot R_\eta(\alpha(z)) + \theta(x \cdot R_\lambda(y) + \theta x \cdot y) \cdot \alpha(z) \\
&= \alpha(x) \cdot (R_\lambda(y) \cdot R_\eta(z)) + \theta\alpha(x) \cdot (y \cdot R_\eta(z) + R_\lambda(y) \cdot z + \theta y \cdot z) \\
&\text{(Hom-associativity)} \\
&= \alpha(x) \cdot R_{\lambda\eta}(y \cdot R_\eta(z) + R_\lambda(y) \cdot z + \theta y \cdot z) \\
&+ \theta\alpha(x) \cdot (y \cdot R_\eta(z) + R_\lambda(y) \cdot z + \theta y \cdot z) = \alpha(x) \prec_{\lambda\eta} (y \prec_\eta z + y \succ_\lambda z).
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
&(x \succ_\lambda y) \prec_\eta \alpha(z) = (R_\lambda(x) \cdot y) \prec_\eta \alpha(z) \\
&= (R_\lambda(x) \cdot y) \cdot R_\eta(\alpha(z)) + \theta(R_\lambda(x) \cdot y) \cdot \alpha(z) \\
&= R_\lambda(\alpha(x)) \cdot (y \cdot R_\eta(z) + \theta y \cdot z) (\text{ by multiplicativity and Hom-associativity}) \\
&= R_\lambda(\alpha(x)) \cdot (y \prec_\eta z) = \alpha(x) \succ_\lambda (y \prec_\eta z).
\end{aligned}$$

Finally we get, using the multiplicativity, the Hom-associativity and (2.4):

$$\begin{aligned}
&\alpha(x) \succ_\lambda (y \succ_\eta z) = \alpha(x) \succ_\lambda (R_\eta(y) \cdot z) = R_\lambda(\alpha(x)) \cdot (R_\eta(y) \cdot z) \\
&= (R_\lambda(x) \cdot R_\eta(y)) \cdot \alpha(z) = R_{\lambda\eta}(R_\lambda(x) \cdot y + x \cdot R_\eta(y) + \theta x \cdot y) \cdot \alpha(z) \\
&= (R_\lambda(x) \cdot y + x \cdot R_\eta(y) + \theta x \cdot y) \succ_{\lambda\eta} \alpha(z) = (x \succ_\lambda y + x \prec_\eta y) \succ_{\lambda\eta} \alpha(z).
\end{aligned}$$

□

Proposition 3.6. *Let $(D, \{\prec_\lambda, \succ_\lambda, \lambda \in \Omega\}, \alpha)$ be a Hom-dendriform family algebra. Then $(D \otimes \mathbb{K}\Omega, \prec, \succ, \tilde{\alpha})$ is Hom-dendriform algebra where for all $\lambda, \eta \in \Omega$, $x, y \in D$:*

$$(x \otimes \lambda) \prec (y \otimes \eta) := (x \prec_\eta y) \otimes \lambda\eta \text{ and } (x \otimes \lambda) \succ (y \otimes \eta) := (x \succ_\lambda y) \otimes \lambda\eta.$$

Proof. The multiplicativity of $\tilde{\alpha}$ with respect to $\prec$ and $\succ$ follows from the one of α with respect to $\prec_\lambda$ and $\succ_\lambda$ for every $\lambda \in \Omega$. □

Let $x, y, z \in D$ and $\lambda, \eta, \tau \in \Omega$. Then, we compute:

$$((x \otimes \lambda) \prec (y \otimes \eta)) \prec \tilde{\alpha}(z \otimes \tau) = \Big((x \prec_\eta y) \otimes \lambda\eta\Big) \prec (\alpha(z) \otimes \tau)$$
$$= \Big((x \prec_\eta y) \prec_\tau \alpha(z)\Big) \otimes \lambda\eta\tau = \Big(\alpha(x) \prec_{\eta\tau} (y \prec_\tau z + y \succ_\eta z)\Big) \otimes \lambda\eta\tau$$
(by equation (3.14))
$$= (\alpha(x) \otimes \lambda) \prec \Big((y \prec_\tau z + y \succ_\eta z) \otimes \eta\tau\Big)$$
$$= \tilde{\alpha}(x \otimes \lambda) \prec \Big((y \otimes \eta) \prec (z \otimes \tau) + (y \otimes \eta) \succ (z \otimes \tau)\Big).$$

Similarly, we obtain

$$\Big((x \otimes \lambda) \succ (y \otimes \eta)\Big)\tilde{\alpha}(z \otimes \tau) = \Big((x \succ_\lambda y) \otimes \lambda\eta\Big) \prec (\alpha(z) \otimes \tau)$$
$$= \Big((x \succ_\lambda y) \prec_\tau \alpha(z)\Big) \otimes \lambda\eta\tau = \Big(\alpha(x) \succ_\lambda (y \prec_\tau z)\Big) \otimes \lambda\eta\tau \text{ (by equation (3.15)}$$
$$= (\alpha(x) \otimes \lambda) \succ \Big((y \prec_\tau z) \otimes \eta\tau\Big) = \tilde{\alpha}(x \otimes \lambda) \succ \Big((y \otimes \eta) \prec (z \otimes \tau)\Big).$$

Finally, we get

$$\tilde{\alpha}(x \otimes \lambda) \succ \Big((y \otimes \eta) \succ (z \otimes \tau)\Big) = (\alpha(x) \otimes \lambda) \succ \Big((y \succ_\eta z) \otimes \eta\tau\Big)$$
$$= \Big(\alpha(x) \succ_\lambda (y \succ_\eta z)\Big) \otimes \lambda\eta\tau = \Big((x \prec_\eta y + x \succ_\lambda y) \succ_{\lambda\eta} \alpha(z)\Big) \otimes \lambda\eta\tau$$
(by equation (3.16))
$$= \Big((x \prec_\eta y + x \succ_\lambda y) \otimes \lambda\eta\Big) \succ (\alpha(z) \otimes \tau)$$
$$= \Big((x \otimes \lambda) \prec (y \otimes \eta) + (x \otimes \lambda) \succ (y \otimes \eta)\Big) \succ \tilde{\alpha}(z \otimes \tau).$$

Similarly, we introduce Hom-type of the concept of tridendriform family algebras which was introduced in [23] as a generalization of tridendriform algebras invented by Loday and Ronco in the study of polytopes and Koszul duality [19].

Definition 3.7. *A* **Hom-tridendriform family algebra** *is a $\mathbb{K}$-Hom-module (T, α) equipped with a family of binary operations $\{\prec_\lambda, \succ_\lambda, \lambda \in \Omega\}$ and a binary operation $\cdot$ such that for $x, y, z \in T$ and $\lambda, \eta \in \Omega$,*

$$\alpha \circ \prec_\lambda = \prec_\lambda \circ \alpha^{\otimes 2}, \ \alpha \circ \succ_\lambda = \succ_\lambda \circ \alpha^{\otimes 2} \text{ and } \alpha \circ \cdot = \cdot \circ \alpha^{\otimes 2}, \tag{3.17}$$
$$(x \prec_\lambda y) \prec_\eta \alpha(z) = \alpha(x) \prec_{\lambda\eta} (y \prec_\eta z + y \succ_\lambda z + y \cdot z), \tag{3.18}$$
$$(x \succ_\lambda y) \prec_\eta \alpha(z) = \alpha(x) \succ_\lambda (y \prec_\eta z), \tag{3.19}$$
$$(x \prec_\eta y + x \succ_\lambda y + x \cdot y) \succ_{\lambda\eta} \alpha(z) = \alpha(x) \succ_\lambda (y \succ_\eta z), \tag{3.20}$$
$$(x \succ_\lambda y) \cdot \alpha(z) = \alpha(x) \succ_\lambda (y \cdot z), \tag{3.21}$$
$$(x \prec_\lambda y) \cdot \alpha(z) = \alpha(x) \cdot (y \succ_\lambda z), \tag{3.22}$$
$$(x \cdot y) \prec_\lambda \alpha(z) = \alpha(x) \cdot (y \prec_\lambda z), \tag{3.23}$$
$$(x \cdot y) \cdot \alpha(z) = \alpha(x) \cdot (y \cdot z). \tag{3.24}$$

Remark. (1) *As in Remark 3, when the semigroup Ω is taken to be the trivial monoid with one single element, a Hom-tridendriform family algebra is precisely a Hom-tridendriform algebra. Furtheremore, a Hom-tridendriform family algebra reduces to a tridendriform family algebra when $\alpha = Id$.*

(2) *Any Hom-tridendriform family algebra gives a Hom-dendriform family algebra by setting $x \cdot y = 0$ for all $x, y \in T$.*

We can prove easily the following result which gives a method for constructing Hom-(tri)dendriform family algebras from (tri)dendriform family algebras.

Theorem 3.8. *Let $(A, \{\prec_\lambda, \succ_\lambda, \lambda \in \Omega\}, \cdot)$ be a tridendriform family algebra and $\alpha : A \to A$ be a self-morphism of tridendriform family algebra. Then $(A, \{\prec^\alpha_\lambda, \succ^\alpha_\lambda, \lambda \in \Omega\}, \cdot_\alpha, \alpha)$ is Hom-tridendriform family algebra where*

$$x \prec^\alpha_\lambda y := \alpha(x \prec_\lambda y),\ x \succ^\alpha_\lambda y := \alpha(x \succ_\lambda y) \text{ and } x \cdot_\alpha y := \alpha(x \cdot y)$$
$$\text{for all } \lambda \in \Omega \text{ and } x, y \in A.$$

Proof. Straightforward computations. □

Proposition 3.9. *A Hom-tridendriform family algebra $(T, \{\prec_\lambda, \succ_\lambda, \lambda \in \Omega\}, \cdot, \alpha)$ induces a Hom-dendriform family algebra $(T, \{\prec'_\lambda, \succ'_\lambda, \lambda \in \Omega\}, \alpha)$ where*

$$a \prec'_\lambda b := a \prec_\lambda b + a \cdot b \text{ and } a \succ'_\lambda b := a \succ_\lambda b \text{ for all } \lambda \in \Omega \text{ and } a, b \in A.$$

Proof. First, the multiplicativity of α with respect to $\cdot$, $\prec_\lambda$ and $\succ_\lambda$ imply the one of α with respect to $\prec'_\lambda$ and $\succ'_\lambda$ for any $\lambda \in \Omega$. Next, for all $x, y, z \in T$ and $\lambda, \eta \in \Omega$, we compute

$$\begin{aligned}
&(x \prec'_\lambda y) \prec'_\eta \alpha(z) = (x \prec_\lambda y) \prec_\eta \alpha(z) + (x \cdot y) \prec_\eta \alpha(z) + (x \prec_\lambda y) \cdot \alpha(z) \\
&+(x \cdot y) \cdot \alpha(z) = \alpha(x) \prec_{\lambda\eta} (y + \prec_\eta z + y \succ_\lambda z + y \cdot z) + \alpha(x) \cdot (y \prec_\eta z) \\
&+\alpha(x) \cdot (y \succ_\lambda z) + \alpha(x) \cdot (y \cdot z) \text{ (by eqautions (3.18), (3.23), (3.22) and (3.24))} \\
&= \alpha(x) \prec_{\lambda\eta} (y \prec_\eta z + y \succ_\lambda z + y \cdot z) + \alpha(x) \cdot (y \prec_\eta z + y \succ_\lambda z + y \cdot z) \\
&= \alpha(x) \prec_{\lambda\eta} (y \prec'_\eta z + y \succ'_\lambda z) + \alpha(x) \cdot (y \prec'_\eta z + y \succ'_\lambda z) \\
&= \alpha(x) \prec'_{\lambda\eta} (y \prec'_\eta z + y \succ'_\lambda z).
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
&(x \succ'_\lambda y) \prec'_\eta \alpha(z) = (x \succ'_\lambda y) \prec_\eta \alpha(z) + (x \succ'_\lambda y) \cdot \alpha(z) \\
&= (x \succ_\lambda y) \prec_\eta \alpha(z) + (x \succ_\lambda y) \cdot \alpha(z) = \alpha(x) \succ_\lambda (y \prec_\eta z) + \alpha(x) \succ_\lambda (y \cdot z) \\
&\text{ (by (3.19) and (3.22))} \\
&= \alpha(x) \succ_\lambda (y \prec'_\eta z) = \alpha(x) \succ'_\lambda (y \prec'_\eta z).
\end{aligned}$$

Finally, we get by (3.20) :

$$\begin{aligned}
&\alpha(x) \succ'_\lambda (y \succ'_\eta z) = \alpha(x) \succ_\lambda (y \succ_\eta z) = (x \prec_\eta y + x \succ_\lambda y + x \cdot y) \succ_{\lambda\eta} \alpha(z) \\
&= (x \succ'_\lambda y + x \prec'_\eta y) \succ_{\lambda\eta} \alpha(z) = (x \succ'_\lambda y + x \prec'_\eta y) \succ'_{\lambda\eta} \alpha(z).
\end{aligned}$$

□

Proposition 3.10. *Let $(T, \{\prec_\lambda, \succ_\lambda, \lambda \in \Omega\}, \cdot, \alpha)$ be a Hom-tridendriform family algebra. Then $(T \otimes \mathbb{K}\Omega, \prec, \succ, \bullet, \tilde{\alpha})$ is Hom-tridendriform algebra where for all $\lambda, \eta \in \Omega$, $x, y \in T$:*

$$(x \otimes \lambda) \prec (y \otimes \eta) := (x \prec_\eta y) \otimes \lambda\eta,\ (x \otimes \lambda) \succ (y \otimes \eta) := (x \succ_\lambda y) \otimes \lambda\eta$$
$$\text{and } (x \otimes \lambda) \bullet (y \otimes \eta) := (x \cdot y) \otimes \lambda\eta.$$

Proof. The multiplicativity of $\tilde{\alpha}$ with respect to $\prec, \succ$ and $\bullet$ follows from the one of α with respect to $\prec_\lambda, \succ_\lambda$ and $\cdot$ for each $\lambda \in \Omega$. Now, let $x, y, z \in T$ and $\lambda, \eta, \tau \in \Omega$. Then, we have first:

$$\Big((x\otimes\lambda)\prec(y\otimes\eta)\Big)\prec\tilde{\alpha}(z\otimes\tau)=\Big((x\prec_\eta y)\otimes\lambda\eta\Big)\prec(\alpha(z)\otimes\tau)$$
$$=\Big((x\prec_\eta y)\prec_\tau\alpha(z)\Big)\otimes\lambda\eta\tau=\Big(\alpha(x)\prec_{\eta\tau}(y\prec_\tau z+y\succ_\eta z+y\cdot z)\Big)\otimes\lambda\eta\tau$$
(by (3.18))
$$=\tilde{\alpha}(x\otimes\lambda)\prec\Big((y\prec_\tau z+y\succ_\eta z+y\cdot z)\otimes\eta\tau\Big)$$
$$=\tilde{\alpha}(x\otimes\lambda)\prec\Big((y\prec_\tau z)\otimes\eta\tau+(y\succ_\eta z)\otimes\eta\tau+(y\cdot z)\otimes\eta\tau\Big)$$
$$=\tilde{\alpha}(x\otimes\lambda)\prec\Big((y\otimes\eta)\prec(z\otimes\tau)+(y\otimes\eta)\succ(z\otimes\tau)+(y\otimes\eta)\bullet(z\otimes\tau)\Big),$$

$$\Big((x\otimes\lambda)\succ(y\otimes\eta)\Big)\prec\tilde{\alpha}(z\otimes\tau)=\Big((x\succ_\lambda y)\otimes\lambda\eta\Big)\prec(\alpha(z)\otimes\tau)$$
$$=\Big((x\succ_\lambda y)\prec_\tau\alpha(z)\Big)\otimes\lambda\eta\tau=\Big(\alpha(x)\succ_\lambda(y\prec_\tau z)\Big)\otimes\lambda\eta\tau\ \text{(by (3.19))}$$
$$=(\alpha(x)\otimes\lambda)\succ\Big((y\prec_\tau z)\otimes\eta\tau\Big)=\tilde{\alpha}(x\otimes\lambda)\succ\Big((y\otimes\eta)\prec(z\otimes\tau)\Big),$$

$$\tilde{\alpha}(x\otimes\lambda)\succ\Big((y\otimes\eta)\succ(z\otimes\tau)\Big)=(\alpha(x)\otimes\lambda)\succ\Big((y\succ_\eta z)\otimes\eta\tau\Big)$$
$$=\Big(\alpha(x)\succ_\lambda(y\succ_\eta z)\Big)\otimes\lambda\eta\tau=\Big((x\prec_\eta y+x\succ_\lambda y+x\cdot y)\succ_{\lambda\eta}\alpha(z)\Big)\otimes\lambda\eta\tau$$
(by (3.20))
$$=\Big((x\prec_\eta y+x\succ_\lambda y+x\cdot y)\otimes\lambda\eta\Big)\succ(\alpha(z)\otimes\tau)$$
$$=\Big((x\prec_\eta y)\otimes\lambda\eta+(x\succ_\lambda y)\otimes\lambda\eta+(x\cdot y)\otimes\lambda\eta\Big)\succ\tilde{\alpha}(z\otimes\tau)$$
$$=\Big((x\otimes\lambda)\prec(y\otimes\eta)+(x\otimes\lambda)\succ(y\otimes\eta)+(x\otimes\lambda)\bullet(y\otimes\eta)\Big)\succ\tilde{\alpha}(z\otimes\tau),$$

$$\Big((x\otimes\lambda)\succ(y\otimes\eta)\Big)\bullet\tilde{\alpha}(z\otimes\tau)=\Big((x\succ_\lambda y)\otimes\lambda\eta\Big)\bullet(\alpha(z)\otimes\tau)$$
$$=\Big((x\succ_\lambda y)\cdot\alpha(z)\Big)\otimes\lambda\eta\tau=\Big(\alpha(x)\succ_\lambda(y\cdot z)\Big)\otimes\lambda\eta\tau\ \text{(by (3.21))}$$
$$=(\alpha(x)\otimes\lambda)\succ\Big((y\cdot z)\otimes\eta\tau\Big)=\tilde{\alpha}(x\otimes\lambda)\succ\Big((y\otimes\eta)\bullet(z\otimes\tau)\Big),$$

$$\Big((x\otimes\lambda)\prec(y\otimes\eta)\Big)\bullet\tilde{\alpha}(z\otimes\tau)=\Big((x\prec_\eta y)\otimes\lambda\eta\Big)\bullet(\alpha(z)\otimes\tau)$$
$$=\Big((x\prec_\eta y)\cdot\alpha(z)\Big)\otimes\lambda\eta\tau=\Big(\alpha(x)\cdot(y\succ_\eta z)\Big)\otimes\lambda\eta\tau\ \text{(by (3.22))}$$
$$=(\alpha(x)\otimes\lambda)\bullet\Big((y\succ_\eta z)\otimes\eta\tau\Big)=\tilde{\alpha}(x\otimes\lambda)\bullet\Big((y\otimes\eta)\succ(z\otimes\tau)\Big),$$

$$\Big((x\otimes\lambda)\bullet(y\otimes\eta)\Big)\prec\tilde{\alpha}(z\otimes\tau)=\Big((x\cdot y)\otimes\lambda\eta\Big)\prec(\alpha(z)\otimes\tau)$$
$$=\Big((x\cdot y)\prec_\tau\alpha(z)\Big)\otimes\lambda\eta\tau=\Big(\alpha(x)\cdot(y\prec_\tau z)\Big)\otimes\lambda\eta\tau\ \text{(by (3.23))}$$
$$=(\alpha(x)\otimes\lambda)\bullet\Big((y\prec_\tau z)\otimes\eta\tau\Big)=\tilde{\alpha}(x\otimes\lambda)\bullet\Big((y\otimes\eta)\prec(z\otimes\tau)\Big).$$

Finally, in a similar way, we compute:

$$\Big((x\otimes\lambda)\bullet(y\otimes\eta)\Big)\bullet\tilde{\alpha}(z\otimes\tau)=\Big((x\cdot y)\otimes\lambda\eta\Big)\bullet(\alpha(z)\otimes\tau)$$
$$=\Big((x\cdot y)\cdot\alpha(z)\Big)\otimes\lambda\eta\tau=\Big(\alpha(x)\cdot(y\cdot z)\Big)\otimes\lambda\eta\tau\ (\text{ by (3.24) })$$
$$=(\alpha(x)\otimes\lambda)\bullet\Big((y\cdot z)\otimes\eta\tau\Big)=\tilde{\alpha}(x\otimes\lambda)\bullet\Big((y\otimes\eta)\bullet(z\otimes\tau)\Big).$$

□

Definition 3.11. *A* **left (resp. right) Hom-pre-Lie family algebra** *is a* $\mathbb{K}$-*Hom-module* (L,α) *with a family of binary operations* $\{\rhd_\lambda,\lambda\in\Omega\}$ *(resp.* $\{\lhd_\lambda,\lambda\in\Omega\}$*) such that*

$$\begin{aligned}&\alpha\circ\rhd_\lambda=\rhd_\lambda\circ\alpha^{\otimes 2},\\&\alpha(x)\rhd_\lambda(y\rhd_\eta z)-(x\rhd_\lambda y)\rhd_{\lambda\eta}\alpha(z)\\&=\alpha(y)\rhd_\eta(x\rhd_\lambda z)-(y\rhd_\eta x)\rhd_{\eta\lambda}\alpha(z)\end{aligned}\tag{3.25}$$

(resp.

$$\begin{aligned}&\alpha\circ\lhd_\lambda=\lhd_\lambda\circ\alpha^{\otimes 2}\\&\alpha(x)\lhd_\lambda(y\lhd_\eta z)-(x\lhd_\lambda y)\lhd_{\lambda\eta}\alpha(z)\\&=\alpha(x)\lhd_\eta(z\lhd_\lambda y)-(x\lhd_\eta z)\lhd_{\eta\lambda}\alpha(y)\ )\end{aligned}\tag{3.26}$$

holds for $x,y,z\in A$ *and* $\lambda,\eta\in\Omega$.

Remark.
(1) *When the semigroup* Ω *is taken to be the trivial monoid with one single element, a Hom-pre-Lie family algebra is precisely a Hom-pre-Lie algebra* [21].
(2) *A Hom-pre-Lie family algebra reduces to a pre-Lie family algebra* [25] *when* $\alpha=Id$.

It is easy to prove the following which is a construction of Hom-pre-Lie algebras family from pre-Lie algebras family.

Proposition 3.12. *Let* $\mathcal{A}:=(A,\{\rhd_\lambda,\lambda\in\Omega\})$ *a left (resp. right) pre-Lie family algebra and* α *be a self-morphism of* $\mathcal{A}$. *Then* $\mathcal{A}_\alpha:(A,\{\rhd_\lambda^\alpha,\lambda\in\Omega\},\alpha)$ *is a left (resp. right) Hom-pre-Lie family algebra where for each* $\lambda\in\Omega$,

$$x\rhd_\lambda^\alpha y:=\alpha(x\rhd_\lambda y)\ \textit{for all}\ x,y\in L.$$

In the sequel, left Hom-pre-Lie family algebras will be simply called Hom-pre-Lie family algebras.

Proposition 3.13. *Let* $(D,\{\prec_\lambda,\succ_\lambda,\lambda\in\Omega\},\alpha)$ *be a Hom-dendriform family algebra such that the semigroup* Ω *is commutative. Then* $(D,\{\rhd_\lambda,\lambda\in\Omega\},\alpha)$ *(resp.* $(D,\{\lhd_\lambda,\lambda\in\Omega\},\alpha)$ *is a left ((resp. right) Hom-pre-Lie family algebra where for all* $\lambda,\eta\in\Omega,\ x,y\in D$:

$$x\rhd_\lambda y=x\succ_\lambda y-y\prec_\lambda x\ \textit{and}\ x\lhd_\lambda y=x\prec_\lambda y-y\succ_\lambda x\tag{3.27}$$

Proof. The multiplicativity of α with respect to $\rhd_\lambda$ follows from the one of α with respect to $\prec_\lambda$ and $\succ_\lambda$ for every $\lambda \in \Omega$. First, on the left hand side, we have:

$$\begin{aligned}
& \alpha(x) \rhd_\lambda (y \rhd_\eta z) - (x \rhd_\lambda y) \rhd_{\lambda\eta} \alpha(z) \\
= \quad & \alpha(x) \rhd_\lambda (y \succ_\eta z - z \prec_\eta y) - (x \succ_\lambda y - y \prec_\lambda x) \rhd_{\lambda\eta} \alpha(z) \\
= \quad & \alpha(x) \succ_\lambda (y \succ_\eta z - z \prec_\eta y) - (y \succ_\eta z - z \prec_\eta y) \prec_\lambda \alpha(x) \\
& -\Big((x \succ_\lambda y - y \prec_\lambda x) \succ_{\lambda\eta} \alpha(z) - \alpha(z) \prec_{\lambda\eta} (x \succ_\lambda y - y \prec_\lambda x)\Big) \\
= \quad & \alpha(x) \succ_\lambda (y \succ_\eta z) - \alpha(x) \succ_\lambda (z \prec_\eta y) - (y \succ_\eta z) \prec_\lambda \alpha(x) + (z \prec_\eta y) \prec_\lambda \alpha(x) \\
& -(x \succ_\lambda y) \succ_{\lambda\eta} \alpha(z) + (y \prec_\lambda x) \succ_{\lambda\eta} \alpha(z) + \alpha(z) \prec_{\lambda\eta} (x \succ_\lambda y) - \alpha(z) \prec_{\lambda\eta} (y \prec_\lambda x) \\
= \quad & (x \prec_\eta y) \succ_{\lambda\eta} \alpha(z) + (x \succ_\lambda y) \succ_{\lambda\eta} \alpha(z) - \alpha(x) \succ_\lambda (z \prec_\eta y) - (y \succ_\eta z) \prec_\lambda \alpha(x) \\
& +\alpha(z) \prec_{\eta\lambda} (y \prec_\alpha x) + \alpha(z) \prec_{\eta\lambda} (y \succ_\eta x) - (x \succ_\lambda y) \succ_{\lambda\eta} \alpha(z) \\
& +(y \prec_\lambda x) \succ_{\lambda\eta} \alpha(z) + \alpha(z) \prec_{\lambda\eta} (x \succ_\lambda y) - \alpha(z) \prec_{\lambda\eta} (y \prec_\lambda x) \text{ (by (3.14)-(3.16))} \\
= \quad & (x \prec_\eta y) \succ_{\lambda\eta} \alpha(z) - \alpha(x) \succ_\lambda (z \prec_\eta y) - (y \succ_\eta z) \prec_\lambda \alpha(x) \\
& +\alpha(z) \prec_{\eta\lambda} (y \succ_\eta x) + (y \prec_\lambda x) \succ_{\lambda\eta} \alpha(z) + \alpha(z) \prec_{\lambda\eta} (x \succ_\lambda y).
\end{aligned}$$

Secondely, on the right hand side, we have:

$$\begin{aligned}
& \alpha(y) \rhd_\eta (x \rhd_\lambda z) - (y \rhd_\eta x) \rhd_{\eta\lambda} \alpha(z) \\
= \quad & \alpha(y) \rhd_\eta (x \succ_\lambda z - z \prec_\lambda x) - (y \succ_\eta x - x \prec_\eta y) \rhd_{\eta\lambda} \alpha(z) \\
= \quad & \alpha(y) \succ_\eta (x \succ_\lambda z - z \prec_\lambda x) - (x \succ_\lambda z - z \prec_\lambda x) \prec_\eta \alpha(y) \\
& -\Big((y \succ_\eta x - x \prec_\eta y) \succ_{\eta\lambda} \alpha(z) - \alpha(z) \prec_{\eta\lambda} (y \succ_\eta x - x \prec_\eta y)\Big) \\
= \quad & \alpha(y) \succ_\eta (x \succ_\lambda z) - \alpha(y) \succ_\eta (z \prec_\lambda x) - (x \succ_\lambda z) \prec_\eta \alpha(y) + (z \prec_\lambda x) \prec_\eta \alpha(y) \\
& -(y \succ_\eta x) \succ_{\eta\lambda} \alpha(z) + (x \prec_\eta y) \succ_{\eta\lambda} \alpha(z) + \alpha(z) \prec_{\eta\lambda} (y \succ_\eta x) - \alpha(z) \prec_{\eta\lambda} (x \prec_\eta y) \\
= \quad & (y \succ_\eta x) \succ_{\eta\lambda} \alpha(z) + (y \prec_\lambda x) \succ_{\eta\lambda} \alpha(z) - \alpha(y) \succ_\eta (z \prec_\lambda x) - (x \succ_\lambda z) \prec_\eta \alpha(y) \\
& +\alpha(z) \prec_{\lambda\eta} (x \prec_\eta y) + \alpha(z) \prec_{\lambda\eta} (x \succ_\lambda y) - (y \succ_\eta x) \succ_{\eta\lambda} \alpha(z) \text{ (by (3.14)-(3.16))} \\
& +(x \prec_\eta y) \succ_{\eta\lambda} \alpha(z) + \alpha(z) \prec_{\eta\lambda} (y \succ_\eta x) - \alpha(z) \prec_{\eta\lambda} (x \prec_\eta y) \\
= \quad & (y \prec_\lambda x) \succ_{\eta\lambda} \alpha(z) - \alpha(y) \succ_\eta (z \prec_\lambda x) - (x \succ_\lambda z) \prec_\eta \alpha(y) \\
& +\alpha(z) \prec_{\lambda\eta} (x \succ_\lambda y) + (x \prec_\eta y) \succ_{\eta\lambda} \alpha(z) + \alpha(z) \prec_{\eta\lambda} (y \succ_\eta x).
\end{aligned}$$

Finally, using the commutativity of the semigroup Ω, we observe that the i-th term in the expansion of the left hand side equals to the $\sigma(i)$-th term in the expansion of the right hand side, where σ is the following permutation of order 6:

$$\begin{pmatrix} i \\ \sigma(i) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 6 & 1 & 4 \end{pmatrix}.$$

Hence,

$$\alpha(x) \rhd_\lambda (y \rhd_\eta z) - (x \rhd_\lambda y) \rhd_{\lambda\eta} \alpha(z) = \alpha(y) \rhd_\eta (x \rhd_\lambda z) - (y \rhd_\eta x) \rhd_{\eta\lambda} \alpha(z)$$

i.e., the desired equation holds. Similarly, we prove that $(D, \{\lhd_\lambda, \lambda \in \Omega\}, \alpha)$ is a right Hom-pre-Lie family algebra. ☐

Corollary 3.1. *A Rota-Baxter family $\{R_\lambda, \lambda \in \Omega\}$ of weight $\theta \in \mathbb{K}$ on a Hom-associative algebra $(A, \cdot, \alpha)$ induces a left (resp. a right) Hom-pre-Lie family algebra $(A, \{\rhd_\lambda, \lambda \in \Omega\}, \alpha)$ (resp. $(A, \{\lhd_\lambda, \lambda \in \Omega\}, \alpha)$) where for all $\lambda \in \Omega$ and $a, b \in A$:*

$$a \rhd_\lambda b := R_\lambda(a) \cdot (b) - a \cdot R_\lambda(b) - \theta a \cdot b \text{ and } a \lhd_\lambda b := R_\lambda(a) \cdot (b) - R_\lambda(a) \cdot (b) + \theta a \cdot b.$$

Proof. Follows by Proposition 3.5 and Proposition 3.13 . □

Definition 3.14. *A* **Nigenhuis family** *on a Hom-associative algebra* $(A, \cdot, \alpha)$ *is a collection of linear operators* $\{N_\lambda, \lambda \in \Omega\}$ *on* $(A, \cdot, \alpha)$ *satisfying:*

$$\alpha \circ N_\lambda = N_\lambda \circ \alpha \text{ for all } \lambda \in \Omega, \tag{3.28}$$

$$N_\lambda(a) \cdot N_\eta(b) = N_{\lambda\eta}\Big(N_\lambda(a) \cdot b + a \cdot N_\eta(b) - N_{\lambda\eta}(a \cdot b)\Big) \tag{3.29}$$

for $a, b \in A$ *and* $\lambda, \eta \in \Omega$.

Observe that any Nijenhuis operator N on a Hom-associative algebra $(A, \cdot, \alpha)$ can be viewed as a constant Nijenhuis family $\{N_\lambda, \lambda \in \Omega\}$ where $N_\lambda = N$ for all $\lambda \in \Omega$.

Theorem 3.15. *Let* $\{N_\lambda, \lambda \in \Omega\}$ *be a Nijenhuis family on the Hom-associative algebra* $(A, \cdot, \alpha)$*. Then the map*

$$N : A \otimes \mathbb{K}\Omega \to A \otimes \mathbb{K}\Omega, \ N(a \otimes \lambda) := N_\lambda(a) \otimes \lambda$$

is a Nijenhuis operator on the Hom-associative algebra $(A \otimes \mathbb{K}\Omega, \bullet, \tilde{\alpha})$.

Proof. First, it is clear that $\tilde{\alpha} \circ N = N \circ \tilde{\alpha}$. Next, pick $a, b \in A$ and $\lambda, \eta \in \Omega$. Then, we have:

$$N(a \otimes \lambda) \bullet N(b \otimes \eta) = (N_\lambda(a) \otimes \lambda) \bullet (N_\eta(b) \otimes \eta) = (N_\lambda(a) \cdot N_\eta(b)) \otimes \lambda\eta.$$

On the other hand, we also get:

$$\begin{aligned} &N(N(a \otimes \lambda)(b \otimes \eta) + (a \otimes \lambda)N(b \otimes \eta) - N((a \otimes \lambda) \bullet (b \otimes \eta))) \\ &= N(N_\lambda(a) \cdot b \otimes \lambda\eta + a \cdot N_\eta(b) \otimes \lambda\eta - N_{\lambda\eta}(ab) \otimes \lambda\eta) \\ &= N_{\lambda\eta}\Big(N_\lambda(a) \cdot b + a \cdot N_\eta(b) - N_{\lambda\eta}(a \cdot b)\Big) \otimes \lambda\eta. \end{aligned}$$

Hence, the conclusion follows by (3.29). □

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