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TRANS-SASAKIAN MANIFOLDS WITH SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. In this study, we use the Schouten-Van Kampen connection on a trans-Sasakian manifold of type (α, β) and obtain the curvature tensors of a trans-Sasakian manifold with respect to this connection. We give an example for trans-Sasakian manifolds with Schouten-Van Kampen connection. Also, we study η-Ricci solitons on trans-Sasakian manifolds with Schouten-Van Kampen connection and give some results.

1. Introduction

Trans-Sasakian manifold which is a new class of almost contact metric manifolds has been introduced by J.A. Oubina [21] and later, Blair and Oubina have studied on some fundamental results of this manifold [3]. A trans-Sasakian manifold is said to be of type (α, β) and it is known that, trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic, β-Kenmotsu and α-Sasakian, respectively [16]. The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been characterized by J. C. Marrero [18] and he has proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu.

In [1], [23] and [25], the authors have studied some different classifications of trans-Sasakian manifolds. Some characterizations for 3-dimensional trans-Sasakian manifolds, ϵ-trans-Sasakian manifolds and Lorentzian trans-Sasakian manifolds have been given in [9], [26] and [10], respectively. Also in lightlike geometry, the indefinite trans-Sasakian manifolds have been studied in [17]. Besides, the Ricci soliton which is a generalization of the Einstein metric has been a popular studying area for geometers recently. There are lots of studies about Ricci solitons on contact and paracontact manifold types in the literature, such as [2], [6], [12], [14], [19], [22], [28], [29] and etc. Furthermore, η-Ricci soliton which is a more general notion of Ricci soliton has been introduced [7] and it has been treated on Hopf hypersurfaces in complex space forms [5].

On the other hand, the Schouten–van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection, (see [15], [20], [24]).

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In the present paper, we use the Schouten-Van Kampen connection on a trans-Sasakian manifold of type (α, β) and obtain the expressions of the Riemannian curvature tensor, Ricci tensor and scalar curvature of a trans-Sasakian manifold with respect to the Schouten-Van Kampen connection. We construct an example of a 3-dimensional trans-Sasakian manifold admitting the Schouten-Van Kampen connection. Finally, we give some characterizations for η-Ricci solitons on trans-Sasakian manifolds with Schouten-Van Kampen connection.

2. Preliminaries

Let M be a $(2n+1)$-dimensional **almost contact metric manifold** with an almost contact metric structure (Φ, ξ, η, g), where Φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\Phi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \Phi\xi = 0,\ \eta \circ \Phi = 0, \tag{2.1}$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \Phi Y) = -g(\Phi X, Y),\ g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in \chi(M)$ [4].

An almost contact metric structure (Φ, ξ, η, g) on M is called a **trans-Sasakian structure** [21], if $(M \times \mathbb{R}, J, g)$ belongs to the class W_4 [13], where J is the almost coplex structure on $M \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\Phi X - f\xi, \eta(X)\frac{d}{dt}),$$

for all vector fields X on M and smooth functions f on $M \times \mathbb{R}$. This may be expressed by the condition

$$(\nabla_X \Phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\Phi X, Y)\xi - \eta(Y)\Phi X), \tag{2.4}$$

for some smooth functions α and β on M [3]. Here, we say that the trans-Sasakian structure is **of type** (α, β).

From equation (2.4), it follows that

$$\nabla_X \xi = -\alpha\Phi X + \beta(X - \eta(X)\xi) \tag{2.5}$$

and

$$(\nabla_X \eta)Y = -\alpha g(\Phi X, Y) + \beta g(\Phi X, \Phi Y). \tag{2.6}$$

Also, on a trans-Sasakian manifold M^{2n+1} with structure (Φ, ξ, η, g), we have the following relations [11]:

$$\begin{aligned} R(X, Y)\xi = {} & (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\Phi X - \eta(X)\Phi Y) \\ & + (Y\alpha)\Phi X - (X\alpha)\Phi Y + (Y\beta)\Phi^2 X - (X\beta)\Phi^2 Y, \end{aligned} \tag{2.7}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - (\xi\beta))(\eta(X)\xi - X), \tag{2.8}$$

$$2\alpha\beta + (\xi\alpha) = 0, \tag{2.9}$$

$$S(X, \xi) = (2n(\alpha^2 - \beta^2) - (\xi\beta))\eta(X) - (2n-1)(X\beta) - (\Phi X)\alpha, \tag{2.10}$$

$$Q\xi = (2n(\alpha^2 - \beta^2) - (\xi\beta))\xi - (2n-1)\operatorname{grad}\beta + \Phi\operatorname{grad}\alpha. \tag{2.11}$$

Here, R is Riemannian curvature tensor, S is Ricci tensor defined by $S(X, Y) = g(QX, Y)$ and Q is Ricci operator.

In this study, we use the Schouten-Van Kampen connection $\tilde{\nabla}$ defined by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + ((\nabla_X \eta)Y)\xi, \tag{2.12}$$

for all $X, Y \in \chi(M)$ [27]. Using (2.5) and (2.6) in (2.12), the Schouten-Van Kampen connection $\tilde{\nabla}$ is defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha\{\eta(Y)\Phi X - g(\Phi X, Y)\xi\} - \beta\{\eta(Y)X - g(X, Y)\xi\}, \tag{2.13}$$

for a trans-Sasakian manifold.

So, we obtain the following Proposition by using (2.1)-(2.5) and (2.13):

Proposition 1. *We have the following properties for the Schouten-Van Kampen connection $\tilde{\nabla}$ on a trans-Sasakian manifold:*

$$\tilde{\nabla} g = 0, \ \tilde{\nabla}\eta = 0, \ \tilde{\nabla}\xi = 0 \tag{2.14}$$

and

$$\tilde{\nabla}\Phi = 0. \tag{2.15}$$

Also from (2.13), the torsion tensor of the Schouten-Van Kampen connection $\tilde{\nabla}$ is

$$\begin{aligned} \tilde{T}(X, Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \\ &= \alpha\{\eta(Y)\Phi X - \eta(X)\Phi Y + 2g(X, \Phi Y)\xi\} - \beta\{\eta(Y)X - \eta(X)Y\}, \end{aligned} \tag{2.16}$$

for all $X, Y \in \chi(M)$.

3. Curvature Tensors of a Trans-Sasakian Manifold with respect to Schouten-Van Kampen Connection

In this section, we obtain the Riemannian curvature tensor, Ricci tensor and scalar curvature of a trans-Sasakian manifold with respect to Schouten-Van Kampen connection and give some results.

Let M be a $(2n+1)$-dimensional trans-Sasakian manifold of type (α, β). Then the Riemannian curvature tensor $\tilde{R}$ of M with respect to the Schouten-Van Kampen connection $\tilde{\nabla}$ is defined by

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z, \tag{3.1}$$

for all $X, Y, Z \in \chi(M)$. From (2.1)-(2.5) and (2.13), we have

$$\begin{aligned} \tilde{\nabla}_X\tilde{\nabla}_Y Z = {} & \nabla_X\nabla_Y Z + (X\alpha)\eta(Z)\Phi Y - (X\beta)\eta(Z)Y - (X\alpha)g(\Phi Y, Z)\xi + (X\beta)g(Y, Z)\xi \\ & + \alpha\left\{\begin{array}{c} \eta(\nabla_Y Z)\Phi X + \eta(\nabla_X Z)\Phi Y + \eta(Z)\Phi(\nabla_X Y) \\ -g(\Phi X, \nabla_Y Z)\xi - g(\Phi Y, \nabla_X Z)\xi + g(\Phi Z, \nabla_X Y)\xi \end{array}\right\} \\ & + \beta\left\{\begin{array}{c} -\eta(\nabla_Y Z)X - \eta(\nabla_X Z)Y - \eta(Z)\nabla_X Y \\ +g(X, \nabla_Y Z)\xi + g(Y, \nabla_X Z)\xi + g(Z, \nabla_X Y)\xi \end{array}\right\} \\ & + \alpha^2\left\{\begin{array}{c} -g(X, Y)\eta(Z)\xi + g(X, Z)\eta(Y)\xi - g(Z, \Phi X)\Phi Y \\ -\eta(Y)\eta(Z)X + \eta(X)\eta(Y)\eta(Z)\xi \end{array}\right\} \\ & + \beta^2\{\eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X - g(X, Z)Y - g(X, Y)\eta(Z)\xi\} \\ & + \alpha\beta\left\{\begin{array}{c} g(\Phi X, Z)Y + g(X, Z)\Phi Y + g(\Phi X, Z)\eta(Y)\xi \\ -\eta(X)\eta(Z)\Phi Y - 2\eta(Y)\eta(Z)\Phi X \end{array}\right\} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \tilde{\nabla}_{[X,Y]}Z = {} & \nabla_{[X,Y]}Z + \alpha\{\eta(Z)\Phi([X, Y]) - g(\Phi([X, Y]), Z)\xi\} \\ & - \beta\{\eta(Z)[X, Y] - g([X, Y], Z)\xi\}. \end{aligned} \tag{3.3}$$

Using (3.2) and (3.3) in (3.1), we obtain the Riemannian curvature tensor $\tilde{R}$ of M with respect to the Schouten-Van Kampen connection $\tilde{\nabla}$ as

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & R(X,Y)Z + (X\alpha)\{\eta(Z)\Phi Y - g(\Phi Y,Z)\xi\} - (X\beta)\{\eta(Z)Y - g(Y,Z)\xi\} \\ & - (Y\alpha)\{\eta(Z)\Phi X - g(\Phi X,Z)\xi\} + (Y\beta)\{\eta(Z)X - g(X,Z)\xi\} \\ & + \alpha^2 \left\{ \begin{array}{c} g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi - g(Z,\Phi X)\Phi Y \\ +g(Z,\Phi Y)\Phi X + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \end{array} \right\} \\ & + \beta^2 \{g(Y,Z)X - g(X,Z)Y\} \\ & + \alpha\beta \left\{ \begin{array}{c} g(\Phi X,Z)Y - g(\Phi Y,Z)X + g(X,Z)\Phi Y - g(Y,Z)\Phi X \\ +g(\Phi X,Z)\eta(Y)\xi - g(\Phi Y,Z)\eta(X)\xi \\ +\eta(X)\eta(Z)\Phi Y - \eta(Y)\eta(Z)\Phi X \end{array} \right\}. \end{aligned} \tag{3.4}$$

Here, R is the Riemannian curvature tensor of M with respect to Levi-Civita connection ∇ which is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

Proposition 2. *On a trans-Sasakian manifold, we have*

i) $\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y$
$= (X\alpha)\{\eta(Z)\Phi Y - \eta(Y)\Phi Z - 2g(\Phi Y,Z)\xi\} + (X\beta)\{\eta(Y)Z - \eta(Z)Y\}$
$+(Y\alpha)\{\eta(X)\Phi Z - \eta(Z)\Phi X - 2g(\Phi Z,X)\xi\} + (Y\beta)\{\eta(Z)X - \eta(X)Z\}$
$+(Z\alpha)\{\eta(Y)\Phi X - \eta(X)\Phi Y - 2g(\Phi X,Y)\xi\} + (Z\beta)\{\eta(X)Y - \eta(Y)X\}$
$+2\alpha^2\{g(X,\Phi Z)\Phi Y + g(Z,\Phi Y)\Phi X + g(Y,\Phi X)\Phi Z\}$
$+2\alpha\beta \left\{ \begin{array}{c} g(\Phi X,Z)Y + g(\Phi Y,X)Z + g(\Phi Z,Y)X \\ +g(\Phi X,Z)\eta(Y)\xi + g(\Phi Y,X)\eta(Z)\xi + g(\Phi Z,Y)\eta(X)\xi \end{array} \right\},$

ii) $\tilde{\mathcal{R}}(X,Y,Z,W) + \tilde{\mathcal{R}}(X,Y,W,Z) = 0,$

iii) $\tilde{\mathcal{R}}(X,Y,Z,W) + \tilde{\mathcal{R}}(Y,X,Z,W) = 0,$

iv) $\tilde{\mathcal{R}}(X,Y,Z,W) - \tilde{\mathcal{R}}(Z,W,X,Y)$
$= (X\alpha)\{\eta(Z)g(\Phi Y,W) - g(\Phi Y,Z)\eta(W)\} - (X\beta)\{\eta(Z)g(Y,W) - g(Y,Z)\eta(W)\}$
$-(Y\alpha)\{\eta(Z)g(\Phi X,W) - g(\Phi X,Z)\eta(W)\} + (Y\beta)\{\eta(Z)g(X,W) - g(X,Z)\eta(W)\}$
$-(Z\alpha)\{\eta(X)g(\Phi W,Y) - g(\Phi W,X)\eta(Y)\} + (Z\beta)\{\eta(X)g(W,Y) - g(W,X)\eta(Y)\}$
$+(W\alpha)\{\eta(X)g(\Phi Z,Y) - g(\Phi Z,X)\eta(Y)\} - (W\beta)\{\eta(X)g(Z,Y) - g(Z,X)\eta(Y)\}$
$+2\alpha\beta \left\{ \begin{array}{c} g(\Phi X,Z)\{g(Y,W) + \eta(Y)\eta(W)\} + g(Y,\Phi Z)\{g(X,W) + \eta(X)\eta(W)\} \\ +g(\Phi Y,W)\{g(X,Z) + \eta(X)\eta(Z)\} + g(X,\Phi W)\{g(Y,Z) + \eta(Y)\eta(Z)\} \end{array} \right\}.$

Here, $\tilde{\mathcal{R}}$ is Riemannian Christoffel curvature tensor of type (0,4) on M with respect to Schouten-Van Kampen connection $\tilde{\nabla}$ defined by

$$\tilde{\mathcal{R}}(X,Y,Z,W) = g(\tilde{R}(X,Y)Z,W),$$

for all $X,Y,Z,W \in \chi(M)$.

Proof. From first Bianchi identity, (2.3) and (3.4), we get (i).

If $\mathcal{R}$ is Riemannian Christoffel tensor field of type (0,4) on M with respect to Levi-Civita connection ∇ defined by $\mathcal{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, then it is well known that

$$\left. \begin{array}{c} \mathcal{R}(X,Y,Z,W) + \mathcal{R}(X,Y,W,Z) = 0, \\ \mathcal{R}(X,Y,Z,W) + \mathcal{R}(Y,X,Z,W) = 0, \\ \mathcal{R}(X,Y,Z,W) - \mathcal{R}(Z,W,X,Y) = 0. \end{array} \right\} \tag{3.5}$$

So using (2.3), (3.4) and (3.5), we obtain (ii)-(iv). □

So, we can give the following Proposition:

Proposition 3. *On a trans-Sasakian manifold, we have*

$$\tilde{R}(X,Y)\xi = \tilde{R}(\xi,X)\xi = 0, \tag{3.6}$$

$$\begin{aligned}\tilde{R}(\xi,X)Y &= 2\alpha\beta\left\{\eta(Y)\Phi X - g(\Phi X,Y)\xi\right\} - (\xi\beta)\left\{\eta(Y)X - g(X,Y)\xi\right\} \\ &\quad + (Y\alpha)\Phi X - (Y\beta)\Phi^2 X + g(\Phi Y,X)\operatorname{grad}\alpha - g(\Phi X,\Phi Y)\operatorname{grad}\beta,\end{aligned} \tag{3.7}$$

$$\eta(\tilde{R}(X,Y)Z) = g(\tilde{R}(X,Y)Z,\xi) = 0, \tag{3.8}$$

for all $X,Y,Z \in \chi(M)$.

Proof. Using (2.1)-(2.3) and (2.7) in (3.4), we obtain (3.6).

From (2.1)-(2.3) and (2.7), using $(X\alpha) = g(\operatorname{grad}\alpha, X)$, we get

$$\begin{aligned}R(\xi,X)Y &= (\alpha^2-\beta^2)\left\{g(X,Y)\xi - \eta(Y)X\right\} + 2\alpha\beta\left\{g(\Phi Y,X)\xi + \eta(Y)\Phi X\right\} \\ &\quad + g(\Phi Y,X)\operatorname{grad}\alpha + (Y\alpha)\Phi X - g(\Phi X,\Phi Y)\operatorname{grad}\beta - (Y\beta)\Phi^2 X\end{aligned} \tag{3.9}$$

on a trans-Sasakian manifold. So, using (2.9) and (3.9) in (3.4), we have (3.7). And finally, from the Proposition 2-(ii) and (3.6), we have (3.8). □

Now, let us obtain the Ricci tensor $\tilde{S}$ and scalar curvature $\tilde{\tau}$ of M with respect to the Schouten-Van Kampen connection $\tilde{\nabla}$.

Proposition 4. *Let M be a $(2n+1)$-dimensional trans-Sasakian manifold of type (α,β). Then, the Ricci tensor $\tilde{S}$ and scalar curvature $\tilde{\tau}$ of M with respect to the Schouten-Van Kampen connection $\tilde{\nabla}$ is*

$$\begin{aligned}\tilde{S}(X,Y) &= S(X,Y) - (2n-2)\alpha\beta g(\Phi X,Y) + \left\{(\xi\beta) + 2n\beta^2\right\} g(X,Y) \\ &\quad - 2n\alpha^2\eta(X)\eta(Y) + \left\{(\Phi X)\alpha + (2n-1)(X\beta)\right\}\eta(Y)\end{aligned} \tag{3.10}$$

and

$$\tilde{\tau} = \tau + 2n\left\{2(\xi\beta) - \alpha^2 + (2n+1)\beta^2\right\}, \tag{3.11}$$

respectively. Also, the Ricci operator $\tilde{Q}$ of M with respect to the Schouten-Van Kampen connection $\tilde{\nabla}$ is

$$\begin{aligned}\tilde{Q}X &= QX - (2n-2)\alpha\beta\Phi X + \left\{(\xi\beta) + 2n\beta^2\right\} X \\ &\quad - 2n\alpha^2\eta(X)\xi + \left\{(\Phi X)\alpha + (2n-1)(X\beta)\right\}\xi,\end{aligned} \tag{3.12}$$

where $\tilde{S}(X,Y) = g(\tilde{Q}X,Y)$. Here, S, Q and τ are Ricci tensor, Ricci operator and scalar curvature of M with respect to the Levi-Civita connection ∇, respectively and $S(X,Y) = g(QX,Y)$, for all $X,Y \in \chi(M)$.

Proof. Using (2.1)-(2.3), (2.9) and (3.4) in

$$\tilde{S}(X,Y) = \sum_{i=1}^{2n+1} g(\tilde{R}(X,e_i)e_i,Y),$$

we get (3.10), where $\{e_i\}, (i = 1,2,...,2n+1)$ is a local orthonormal basis of the trans-Sasakian manifold M. Using (3.10) in

$$\tilde{\tau} = \sum_{i=1}^{2n+1} \tilde{S}(e_i,e_i),$$

we have (3.11). (3.12) is obvious from the definition of Ricci operator and (3.10). □

Corollary 1. *Let M be a $(2n+1)$-dimensional trans-Sasakian manifold of type (α,β). Then, we have*

$$\tilde{S}(X,\xi)=0, \tag{3.13}$$

$$\tilde{S}(\xi,X)=(2n-1)\left\{(\xi\beta)\eta(X)-(X\beta)\right\}-(\Phi X)\alpha \tag{3.14}$$

and

$$\tilde{Q}\xi=(2n-1)\left\{(\xi\beta)\xi-\operatorname{grad}\beta\right\}+\Phi(\operatorname{grad}\alpha), \tag{3.15}$$

for all $X\in\chi(M)$.

Proof. The proof is obvious from (2.1)-(2.3), (2.10), (3.10) and (3.12). □

If the condition of

$$(2n-1)\operatorname{grad}\beta=\Phi(\operatorname{grad}\alpha) \tag{3.16}$$

satisfies on the trans-Sasakian manifold M^{2n+1}, then we have $\xi\beta=0$. So, under the condition (3.16), the equations (3.7), (3.10)-(3.15) reduce to

$$\begin{aligned}\tilde{R}(\xi,X)Y=2\alpha\beta\left\{\eta(Y)\Phi X-g(\Phi X,Y)\xi\right\}+(Y\alpha)\Phi X-(Y\beta)\Phi^2X\\ +g(\Phi Y,X)\operatorname{grad}\alpha-g(\Phi X,\Phi Y)\operatorname{grad}\beta,\end{aligned} \tag{3.17}$$

$$\tilde{S}(X,Y)=S(X,Y)-(2n-2)\alpha\beta g(\Phi X,Y)+2n\beta^2g(X,Y)-2n\alpha^2\eta(X)\eta(Y), \tag{3.18}$$

$$\tilde{\tau}=\tau+2n\left\{-\alpha^2+(2n+1)\beta^2\right\}, \tag{3.19}$$

$$\tilde{Q}X=QX-(2n-2)\alpha\beta\Phi X+2n\beta^2X-2n\alpha^2\eta(X)\xi \tag{3.20}$$

and

$$\tilde{S}(X,\xi)=\tilde{S}(\xi,X)=\tilde{Q}\xi=0. \tag{3.21}$$

From (3.18), we can state the following Theorem:

Theorem 1. *Let M be a 3-dimensional trans-Sasakian manifold of type (α,β). Then, the Ricci tensor $\tilde{S}$ of M with respect to the Schouten-Van Kampen connection $\tilde{\nabla}$ is symmetric if the condition (3.16) satisfies.*

Also, under the condition (3.16), in 3-dimensional trans-Sasakian manifold of type (α,β) we have

$$\tilde{S}(X,Y)=S(X,Y)+2(\beta^2g(X,Y)-\alpha^2\eta(X)\eta(Y)), \tag{3.22}$$

$$\tilde{Q}X=QX+2(\beta^2X-\alpha^2\eta(X)\xi) \tag{3.23}$$

and

$$\tilde{\tau}=\tau+2\left\{3\beta^2-\alpha^2\right\}. \tag{3.24}$$

Here, let us construct an example of a 3-dimensional trans-Sasakian manifold admitting the Schouten-Van Kampen connection.

Example 1. *Consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3,\ z\neq 0\}$, where (x,y,z) are the standart coordinates in $\mathbb{R}^3$.*

We choose the vector fields $\{e_1,e_2,e_3\}$ as

$$e_1=-z\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\right),\ e_2=-z\frac{\partial}{\partial y},\ e_3=\frac{\partial}{\partial z}, \tag{3.25}$$

which are linearly independent at each point of M.

Let g be the Riemannian metric defined by $g(e_i,e_j)=0$, $i\neq j$, $i,j=1,2,3$ and $g(e_k,e_k)=1$, $k=1,2,3$.

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = e_2,\ \phi e_2 = -e_1,\ \phi e_3 = 0. \tag{3.26}$$

Using the linearity of ϕ and g, we have $\eta(e_3) = 1$, $\phi^2 Z = -Z + \eta(Z)e_3$ and $g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U)$, for any $U, Z \in \chi(M)$. Thus, for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Now, we have

$$[e_1, e_2] = -ye_2 - z^2 e_3,\ [e_1, e_3] = -\frac{1}{z}e_1,\ [e_2, e_3] = -\frac{1}{z}e_2. \tag{3.27}$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is defined as

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Taking $e_3 = \xi$ and using Koszul's formula, we get the following

$$\begin{aligned} &\nabla_{e_1} e_1 = \frac{1}{z}e_3,\ \nabla_{e_1} e_2 = -\frac{1}{2}z^2 e_3,\ \nabla_{e_1} e_3 = -\frac{1}{z}e_1 + \frac{1}{2}z^2 e_2, \\ &\nabla_{e_2} e_1 = ye_2 + \frac{1}{2}z^2 e_3,\ \nabla_{e_2} e_2 = -ye_1 + \frac{1}{z}e_3,\ \nabla_{e_2} e_3 = -\frac{1}{2}z^2 e_1 - \frac{1}{z}e_2, \\ &\nabla_{e_3} e_1 = \frac{1}{2}z^2 e_2,\ \nabla_{e_3} e_2 = -\frac{1}{2}z^2 e_1, \nabla_{e_3} e_3 = 0. \end{aligned} \tag{3.28}$$

From the above equalities, it can be easily seen that (ϕ, ξ, η, g) is a trans-Sasakian structure on M. Consequently, (M, ϕ, ξ, η, g) is a trans-Sasakian manifold with $\alpha = -\frac{1}{2}z^2 \neq 0$ and $\beta = -\frac{1}{z}$.

Using the above relations in (2.13), we obtain

$$\begin{aligned} &\tilde{\nabla}_{e_1} e_1 = \tilde{\nabla}_{e_1} e_2 = \tilde{\nabla}_{e_1} e_3 = 0, \\ &\tilde{\nabla}_{e_2} e_1 = ye_2,\ \tilde{\nabla}_{e_2} e_2 = -ye_1,\ \tilde{\nabla}_{e_2} e_3 = 0, \\ &\tilde{\nabla}_{e_3} e_1 = \frac{1}{2}z^2 e_2,\ \tilde{\nabla}_{e_3} e_2 = -\frac{1}{2}z^2 e_1,\ \tilde{\nabla}_{e_3} e_3 = 0. \end{aligned} \tag{3.29}$$

From the above equalities, we can obtain the components of the curvature tensor with respect to the Levi-Civita connection are as follows:

$$\begin{aligned} &R(e_1, e_2)e_1 = \left(y^2 + \frac{1}{z^2} + \frac{3}{4}z^4\right)e_2 - yz^2 e_3, \\ &R(e_1, e_2)e_2 = -\left(y^2 + \frac{1}{z^2} + \frac{3}{4}z^4\right)e_1 + \frac{y}{z}e_3,\ R(e_1, e_2)e_3 = yz^2 e_1 - \frac{y}{z}e_2, \\ &R(e_1, e_3)e_1 = -yz^2 e_2 + \left(\frac{2}{z^2} - \frac{1}{4}z^4\right)e_3\ , R(e_1, e_3)e_2 = yz^2 e_1, \\ &R(e_1, e_3)e_3 = \left(\frac{1}{4}z^4 - \frac{2}{z^2}\right)e_1,\ R(e_2, e_3)e_1 = \frac{y}{z}e_2, \\ &R(e_2, e_3)e_2 = -\frac{y}{z}e_1 + \left(\frac{2}{z^2} - \frac{1}{4}z^4\right)e_3,\ R(e_2, e_3)e_3 = \left(\frac{1}{4}z^4 - \frac{2}{z^2}\right)e_2. \end{aligned} \tag{3.30}$$

On the other hand, the components of the curvature tensor with respect to the Schouten-Van Kampen connection are as follows:

$$\begin{aligned}
&\tilde{R}(e_1,e_2)e_1=\left(y^2+\frac{1}{2}z^4\right)e_2,\ \tilde{R}(e_1,e_2)e_2=-\left(y^2+\frac{1}{2}z^4\right)e_1,\ \tilde{R}(e_1,e_2)e_3=0,\\
&\tilde{R}(e_1,e_3)e_1=-yz^2e_2,\ \tilde{R}(e_1,e_3)e_2=yz^2e_1,\ \tilde{R}(e_1,e_3)e_3=0,\\
&\tilde{R}(e_2,e_3)e_1=\frac{y}{z}e_2,\ \tilde{R}(e_2,e_3)e_2=-\frac{y}{z}e_1,\ \tilde{R}(e_2,e_3)e_3=0.
\end{aligned}\tag{3.31}$$

With the help of the equations (3.30) and (3.31), we get the Ricci tensors of Levi-Civita connection and Schouten-Van Kampen connection, respectively, as follows:

$$\begin{aligned}
&S(e_1,e_1)=-\left(y^2+\frac{3}{z^2}+\frac{1}{2}z^4\right),\ S(e_1,e_2)=0,\ S(e_1,e_3)=\frac{y}{z},\\
&S(e_2,e_1)=0,\ S(e_2,e_2)=-\left(y^2+\frac{3}{z^2}+\frac{1}{2}z^4\right),\ S(e_2,e_3)=yz^2,\\
&S(e_3,e_1)=\frac{y}{z},\ S(e_3,e_2)=yz^2,\ S(e_3,e_3)=\left(\frac{1}{2}z^4-\frac{4}{z^2}\right)
\end{aligned}\tag{3.32}$$

and

$$\begin{aligned}
&\tilde{S}(e_1,e_1)=-\left(y^2+\frac{1}{2}z^4\right),\ \tilde{S}(e_1,e_2)=\tilde{S}(e_1,e_3)=0,\\
&\tilde{S}(e_2,e_1)=0,\ \tilde{S}(e_2,e_2)=-\left(y^2+\frac{1}{2}z^4\right),\ \tilde{S}(e_2,e_3)=0,\\
&\tilde{S}(e_3,e_1)=\frac{y}{z},\ \tilde{S}(e_3,e_2)=yz^2,\ \tilde{S}(e_3,e_3)=0.
\end{aligned}\tag{3.33}$$

Therefore, the scalar curvatures of Levi-Civita connection and Schouten-Van Kampen connection are obtained as

$$\tau=\sum_{i=1}^{3}S(e_i,e_i)=-2\left(\frac{1}{4}z^4+\frac{5}{z^2}+y^2\right)\tag{3.34}$$

and

$$\tilde{\tau}=\sum_{i=1}^{3}\tilde{S}(e_i,e_i)=-(2y^2+z^4),\tag{3.35}$$

respectively.

4. η-Ricci Solitons on 3-Dimensional Trans-Sasakian Manifolds with Schouten-Van Kampen Connection

Throughout this section, we assume that the condition (3.16) is satisfied on 3-dimensional trans-Sasakian manifold.

Let (M,Φ,η,ξ,g) be a 3-dimensional trans-Sasakian manifold. The data (g,V,λ,μ) which satisfy the equation

$$L_Vg+2S+2\lambda g+2\mu\eta\otimes\eta=0\tag{4.1}$$

is called an *η-Ricci soliton* on M [7]. Here, L_V is the Lie derivative operator along the vector field V, S is the Ricci curvature tensor field with respect to Levi-Civita connection and λ and μ are real constants. Especially, if $\mu=0$, then (g,V,λ) is a *Ricci soliton* and it is called *shrinking*, *steady* or *expanding* according to λ is negative, zero or positive, respectively [8].

Let (g, V, λ, μ) be an η-Ricci soliton on a trans-Sasakian manifold with respect to Schouten-Van Kampen connection. Then we have

$$(\tilde{L}_V g)(Y, Z) + 2\tilde{S}(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0, \tag{4.2}$$

where $\tilde{L}_V$ is the Lie derivative along the vector field V on M and $\tilde{S}$ is the Ricci curvature tensor field with respect to the Schouten-Van Kampen connection $\tilde{\nabla}$.

From (2.13), we have

$$\begin{aligned}(\tilde{L}_V g)(Y, Z) &= g(\tilde{\nabla}_Y V, Z) + g(Y, \tilde{\nabla}_Z V)\\ &= (L_V g)(Y, Z) - \alpha g(V, \eta(Z)\Phi Y + \eta(Y)\Phi Z)\\ &\quad + \beta g(\eta(Z)Y + \eta(Y)Z - 2g(Y, Z)\xi, V).\end{aligned} \tag{4.3}$$

So, using (3.22) and (4.3) in (4.2), we obtain that

$$\begin{aligned}&(L_V g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z)\\ &- \alpha\{g(V, \eta(Z)\Phi Y + \eta(Y)\Phi Z) + 4\alpha\eta(Y)\eta(Z)\}\\ &+ \beta\{g(\eta(Z)Y + \eta(Y)Z - 2g(Y, Z)\xi, V) + 4\beta g(Y, Z)\} = 0.\end{aligned} \tag{4.4}$$

Hence, we can state the following Theorem:

Theorem 2. *An η-Ricci soliton (g, V, λ, μ) on a trans-Sasakian manifold is invariant under Schouten-Van Kampen connection if and only if the relation*

$$\begin{aligned}&\alpha\{g(V, \eta(Z)\Phi Y + \eta(Y)\Phi Z) + 4\alpha\eta(Y)\eta(Z)\}\\ &= \beta\{g(\eta(Z)Y + \eta(Y)Z - 2g(Y, Z)\xi, V) + 4\beta g(Y, Z)\}\end{aligned} \tag{4.5}$$

holds for all vector fields Y, Z and V.

Now, let V be pointwise collinear with ξ, i.e. $V = b\xi$, where b is a function on the 3-dimensional trans-Sasakian manifold. Then $(\tilde{L}_V g + 2\tilde{S} + 2\lambda g + 2\mu\eta \otimes \eta)(Y, Z) = 0$ implies

$$g(\tilde{\nabla}_Y b\xi, Z) + g(Y, \tilde{\nabla}_Z b\xi) + 2\tilde{S}(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0 \tag{4.6}$$

and using (2.14) and (3.22) in (4.6), we have

$$\begin{aligned}&(Yb)\eta(Z) + (Zb)\eta(Y) + 2S(Y, Z) + 4\beta^2 g(Y, Z)\\ &- 4\alpha^2\eta(Y)\eta(Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0.\end{aligned} \tag{4.7}$$

Taking $Y = \xi$ in (4.7) and using (2.10), we get

$$(\xi b)\eta(Z) + (Zb) + 2(\lambda + \mu)\eta(Z) = 0. \tag{4.8}$$

Again taking $Z = \xi$ in (4.8), we have $\xi b = -(\lambda + \mu)$ and using this expression in (4.8) we get

$$db = -(\lambda + \mu)\eta, \tag{4.9}$$

where d is the differential operator. Applying d to (4.9), we have $(\lambda + \mu)d\eta = 0$. Since $d\eta \neq 0$, we get $\lambda + \mu = 0$. Using this equation in (4.9), we obtain that b is a constant.

So, from (4.7) we have

$$S(Y, Z) = -(2\beta^2 + \lambda)g(Y, Z) + (2\alpha^2 - \mu)\eta(Y)\eta(Z). \tag{4.10}$$

Thus, we have

Theorem 3. *If (g, V, λ, μ) is an η-Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection and V is pointwise collinear with ξ, then V is a constant multiple of ξ and M is an η-Einstein manifold with respect to Levi-Civita connection.*

Also, from (4.10) we can state the following Corollary:

Corollary 2. *If (g, V, λ, μ) is an η-Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection and V is pointwise collinear with ξ, then V is a constant multiple of ξ and M is of constant scalar curvature provided α and β are constants.*

Here, let (g, ξ, λ, μ) be an η-Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection. Taking $V = \xi$ in (4.2), then we have

$$(\tilde{L}_\xi g)(Y, Z) + 2\tilde{S}(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0. \tag{4.11}$$

Using (2.14) and (3.22) in (4.11), we have

$$S(Y, Z) = -(2\beta^2 + \lambda)g(Y, Z) + (2\alpha^2 - \mu)\eta(Y)\eta(Z). \tag{4.12}$$

So, we have

Theorem 4. *If (g, V, λ, μ) is an η-Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection, then M is an η-Einstein manifold with respect to Levi-Civita connection.*

Taking $Z = \xi$ in (4.12) and using (2.10), we get

$$\lambda + \mu = 0. \tag{4.13}$$

Hence,

Theorem 5. *If (g, V, λ, μ) is an η-Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection, then $\lambda = -\mu$.*

Finally, let us give some results for Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection.

If (g, V, λ) is a Ricci soliton on a trans-Sasakian manifold with respect to Schouten-Van Kampen connection, then taking $\mu = 0$ in (4.2) we have

$$(\tilde{L}_V g)(Y, Z) + 2\tilde{S}(Y, Z) + 2\lambda g(Y, Z) = 0. \tag{4.14}$$

Using (3.22) and (4.3) in (4.14), we have

$$\begin{aligned}
&(L_V g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) \\
&\quad - \alpha\{g(V, \eta(Z)\Phi Y + \eta(Y)\Phi Z) + 4\alpha\eta(Y)\eta(Z)\} \\
&\quad + \beta\{g(\eta(Z)Y + \eta(Y)Z - 2g(Y, Z)\xi, V) + 4\beta g(Y, Z)\} = 0.
\end{aligned} \tag{4.15}$$

So, we can state the following Theorem:

Theorem 6. *A Ricci soliton (g, V, λ) on a trans-Sasakian manifold is invariant under Schouten-Van Kampen connection if and only if the relation (4.5) holds for all vector fields Y, Z and V.*

If V is pointwise collinear with ξ, then taking $\mu = 0$ in (4.7)-(4.9) we have b is constant and the equation (4.10) reduces to

$$S(Y,Z) = -2\beta^2 g(Y,Z) + 2\alpha^2\eta(Y)\eta(Z). \tag{4.16}$$

So, we have

Theorem 7. *If (g, V, λ) is a Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection and V is pointwise collinear with ξ, then V is a constant multiple of ξ and M is an η-Einstein manifold with respect to Levi-Civita connection.*

Corollary 3. *If (g, V, λ) is a Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection and V is pointwise collinear with ξ, then the scalar curvature of M with respect to Levi-Civita connection is $\tau = 2(\alpha^2 - 3\beta^2)$. Also, using this and (4.16) in (3.22)-(3.24), we have $\tilde{S} = \tilde{Q} = \tilde{\tau} = 0$.*

From (4.16), we can state the following Corollary:

Corollary 4. *If (g, V, λ) is a Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection and V is pointwise collinear with ξ, then V is a constant multiple of ξ and M is of constant scalar curvature provided α and β are constants.*

Now, if (g, ξ, λ) is a Ricci soliton on a trans-Sasakian manifold with respect to Schouten-Van Kampen connection, then taking $\mu = 0$ in (4.11), the equation (4.12) reduces to

$$S(Y,Z) = -(2\beta^2 + \lambda)g(Y,Z) + 2\alpha^2\eta(Y)\eta(Z). \tag{4.17}$$

Thus, we get

Theorem 8. *If (g, ξ, λ) is a Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection, then M is an η-Einstein manifold with respect to Levi-Civita connection.*

Taking $Z = \xi$ in (4.17) and using (2.10), we have $\lambda = 0$. Hence,

Theorem 9. *A Ricci soliton (g, ξ, λ) on a trans-Sasakian manifold with respect to Schouten-Van Kampen connection is always steady.*

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