TRANS-SASAKIAN MANIFOLDS WITH SCHOUTEN-VAN KAMPEN CONNECTION

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ABSTRACT. In this study, we use the Schouten-Van Kampen connection on a trans-Sasakian manifold of type $(\alpha, \beta)$ and obtain the curvature tensors of a trans-Sasakian manifold with respect to this connection. We give an example for trans-Sasakian manifolds with Schouten-Van Kampen connection. Also, we study $\eta$-Ricci solitons on trans-Sasakian manifolds with Schouten-Van Kampen connection and give some results.

1. Introduction

Trans-Sasakian manifold which is a new class of almost contact metric manifolds has been introduced by J.A. Oubina [21] and later, Blair and Oubina have studied on some fundamental results of this manifold [3]. A trans-Sasakian manifold is said to be of type $(\alpha, \beta)$ and it is known that, trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic, $\beta$-Kenmotsu and $\alpha$-Sasakian, respectively [16]. The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been characterized by J. C. Marrero [18] and he has proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or $\alpha$-Sasakian or $\beta$-Kenmotsu.

In [1], [23] and [25], the authors have studied some different classifications of trans-Sasakian manifolds. Some characterizations for 3-dimensional trans-Sasakian manifolds, $\epsilon$-trans-Sasakian manifolds and Lorentzian trans-Sasakian manifolds have been given in [9], [26] and [10], respectively. Also in lightlike geometry, the indefinite trans-Sasakian manifolds have been studied in [17]. Besides, the Ricci soliton which is a generalization of the Einstein metric has been a popular studying area for geometers recently. There are lots of studies about Ricci solitons on contact and parac Kahler manifolds types in the literature, such as [2], [6], [12], [14], [19], [22], [28], [29] and etc. Furthermore, $\eta$-Ricci soliton which is a more general notion of Ricci soliton has been introduced [7] and it has been treated on Hopf hypersurfaces in complex space forms [5].

On the other hand, the Schouten–van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection, (see [15], [20], [24]).

2000 Mathematics Subject Classification. 53C15, 53C25, 53D10.

Key words and phrases. Trans-Sasakian Manifold; Schouten-Van Kampen Connection; $\eta$-Ricci Soliton.
In the present paper, we use the Schouten-Van Kampen connection on a trans-Sasakian manifold of type $(\alpha, \beta)$ and obtain the expressions of the Riemannian curvature tensor, Ricci tensor and scalar curvature of a trans-Sasakian manifold with respect to the Schouten-Van Kampen connection. We construct an example of a 3-dimensional trans-Sasakian manifold admitting the Schouten-Van Kampen connection. Finally, we give some characterizations for $\eta$-Ricci solitons on trans-Sasakian manifolds with Schouten-Van Kampen connection.

2. Preliminaries

Let $M$ be a $(2n + 1)$-dimensional almost contact metric manifold with an almost contact metric structure $(\Phi, \xi, \eta, g)$, where $\Phi$ is a $(1, 1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is the compatible Riemannian metric such that

$$\Phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \Phi\xi = 0, \quad \eta \circ \Phi = 0,$$

(2.1)

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

(2.2)

$$g(X, \Phi Y) = -g(\Phi X, Y), \quad g(X, \xi) = \eta(X),$$

(2.3)

for all $X, Y \in \chi(M)$ [4].

An almost contact metric structure $(\Phi, \xi, \eta, g)$ on $M$ is called a trans-Sasakian structure [21], if $(M \times \mathbb{R}, J, g)$ belongs to the class $W_4$ [13], where $J$ is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, f \frac{d}{dt}) = (\Phi X - f\xi, \eta(X) \frac{d}{dt}),$$

for all vector fields $X$ on $M$ and smooth functions $f$ on $M \times \mathbb{R}$. This may be expressed by the condition

$$(\nabla_X \Phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(X, Y)\xi - \eta(Y)\Phi X),$$

(2.4)

for some smooth functions $\alpha$ and $\beta$ on $M$ [3]. Here, we say that the trans-Sasakian structure is of type $(\alpha, \beta)$.

From equation (2.4), it follows that

$$\nabla_X \xi = -\alpha \Phi X + \beta(X - \eta(X)\xi)$$

(2.5)

and

$$(\nabla_X \eta)Y = -\alpha g(\Phi X, Y) + \beta g(\Phi X, \Phi Y).$$

(2.6)

Also, on a trans-Sasakian manifold $M^{2n+1}$ with structure $(\Phi, \xi, \eta, g)$, we have the following relations [11]:

$$R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\Phi X - \eta(X)\Phi Y)$$

$$+ (Y\alpha)\Phi X - (X\alpha)\Phi Y + (Y\beta)\Phi^2 X - (X\beta)\Phi^2 Y,$$

(2.7)

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - (\xi\beta))(\eta(\xi)X - X),$$

(2.8)

$$2\alpha\beta + (\xi\alpha) = 0,$$

(2.9)

$$S(X, \xi) = (2n(\alpha^2 - \beta^2) - (\xi\beta))\eta(X) - (2n - 1)(X\beta) - (\Phi X)\alpha,$$

(2.10)

$$Q\xi = (2n(\alpha^2 - \beta^2) - (\xi\beta))\xi - (2n - 1)\text{grad} \beta + \Phi \text{grad} \alpha.$$  

(2.11)

Here, $R$ is Riemannian curvature tensor, $S$ is Ricci tensor defined by $S(X, Y) = g(QX, Y)$ and $Q$ is Ricci operator.

In this study, we use the Schouten-Van Kampen connection $\tilde{\nabla}$ defined by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + ((\nabla_X \eta)Y)\xi,$$

(2.12)
for all $X, Y \in \chi(M)$. Using (2.5) and (2.6) in (2.12), the Schouten-Van Kampen connection $\hat{\nabla}$ is defined by

$$\hat{\nabla} X Y = \nabla X Y + \alpha \{ \eta(Y) \Phi X - g(\Phi X, Y) \xi \} - \beta \{ \eta(Y) X - g(X, Y) \xi \},$$

(2.13)

for a trans-Sasakian manifold.

So, we obtain the following Proposition by using (2.1)-(2.5) and (2.13):

**Proposition 1.** We have the following properties for the Schouten-Van Kampen connection $\hat{\nabla}$ on a trans-Sasakian manifold:

$$\hat{\nabla} g = 0, \quad \hat{\nabla} \eta = 0, \quad \hat{\nabla} \xi = 0$$

(2.14)

and

$$\hat{\nabla} \Phi = 0.$$  

(2.15)

Also from (2.13), the torsion tensor of the Schouten-Van Kampen connection $\hat{\nabla}$ is

$$\hat{T}(X, Y) = \hat{\nabla} X Y - \hat{\nabla} Y X - [X, Y] = \alpha \{ \eta(Y) \Phi X - \eta(X) \Phi Y + 2g(\Phi Y, Z) \xi \} - \beta \{ \eta(Y) X - \eta(X) Y \},$$

(2.16)

for all $X, Y \in \chi(M)$.

3. Curvature Tensors of a Trans-Sasakian Manifold with respect to Schouten-Van Kampen Connection

In this section, we obtain the Riemannian curvature tensor, Ricci tensor and scalar curvature of a trans-Sasakian manifold with respect to Schouten-Van Kampen connection and give some results.

Let $M$ be a $(2n + 1)$-dimensional trans-Sasakian manifold of type $(\alpha, \beta, \gamma)$. Then the Riemannian curvature tensor $\hat{R}$ of $M$ with respect to the Schouten-Van Kampen connection $\hat{\nabla}$ is defined by

$$\hat{R}(X, Y) Z = \hat{\nabla}_X \hat{\nabla}_Y Z + (X \alpha) \eta(Z) \Phi Y - (X \beta) \eta(Z) Y - (X \alpha) g(\Phi Y, Z) \xi + (X \beta) g(Y, Z) \xi$$

(3.1)

for all $X, Y, Z \in \chi(M)$. From (2.1)-(2.5) and (2.13), we have

$$\hat{\nabla}_X \hat{\nabla}_Y Z = \nabla X \nabla Y Z + (X \alpha) \eta(Z) \Phi Y - (X \beta) \eta(Z) Y - (X \alpha) g(\Phi Y, Z) \xi + (X \beta) g(Y, Z) \xi$$

$$+ \alpha \left\{ \begin{array}{l} \eta(\nabla Y Z) \Phi X + \eta(\nabla X Z) \Phi Y + \eta(Z) \Phi(\nabla_X Y) \\ - g(\Phi X, \nabla Y Z) \xi - g(\Phi Y, \nabla X Z) \xi + g(\Phi Z, \nabla X Y) \xi \end{array} \right\}$$

$$+ \beta \left\{ \begin{array}{l} - \eta(\nabla Y Z) X - \eta(\nabla X Z) Y - \eta(Z) \nabla X Y \\ + g(\Phi X, \nabla Y Z) \xi + g(Y, \nabla X Z) \xi + g(Z, \nabla X Y) \xi \end{array} \right\}$$

$$+ \alpha^2 \left\{ \begin{array}{l} - g(X, Y) \eta(Z) \xi + g(X, Z) \eta(Y) \xi - g(\Phi X, \Phi Y) \\ - \eta(Y) \eta(Z) X + \eta(X) \eta(Y) \eta(Z) \xi \end{array} \right\}$$

$$+ \beta^2 \left\{ \eta(X) \eta(Z) Y + \eta(Y) \eta(Z) X - g(X, Z) Y - g(X, Y) \eta(Z) \xi \right\}$$

$$+ \alpha \beta \left\{ \begin{array}{l} g(\Phi X, Z) Y + g(X, Z) \Phi Y + g(\Phi X, Z) \eta(Y) \xi \\ - \eta(X) \eta(Z) \Phi Y - 2 \eta(Y) \eta(Z) \Phi X \end{array} \right\}$$

(3.2)

and

$$\hat{\nabla}_{[X, Y]} Z = \nabla_{[X, Y]} Z + \alpha \{ \eta(Z) \Phi([X, Y]) - g(\Phi([X, Y]), Z) \xi \}$$

$$- \beta \{ \eta(Z) [X, Y] - g([X, Y], Z) \xi \}.$$  

(3.3)
Using (3.2) and (3.3) in (3.1), we obtain the Riemannian curvature tensor $\hat{R}$ of $M$ with respect to the Schouten-Van Kampen connection $\hat{\nabla}$ as

$$\hat{R}(X, Y, Z) = R(X, Y)Z + (X\alpha)\{\eta(Z)\Phi Y - g(\Phi Y, Z)\xi\} - (X\beta)\{\eta(Y)Z - g(Y, Z)\xi\}$$

$$+ (Y\alpha)\{\eta(Z)\Phi X - g(\Phi X, Z)\xi\} + (Y\beta)\{\eta(Z)X - g(X, Z)\xi\}$$

$$+ \alpha^2 \left\{ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi - g(\Phi Y, Z)\eta(X)\xi \right\} + g(Z, \Phi Y)\Phi X + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X$$

$$+ \beta^2 \{(Y, Z)X - g(Y, Z)Y\}$$

$$+ \alpha\beta \left\{ g(\Phi X, Z)Y - g(\Phi Y, Z)X + g(X, Z)\Phi Y - g(Y, Z)\Phi X \right\} + g(\Phi X, Z)\eta(Y)\xi - g(\Phi Y, Z)\eta(X)\xi + \eta(X)\eta(Z)\Phi Y - \eta(Y)\eta(Z)\Phi X$$

(3.4)

Here, $\hat{R}$ is the Riemannian curvature tensor of $M$ with respect to Levi-Civita connection $\nabla$ which is defined by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$.

**Proposition 2.** On a trans-Sasakian manifold, we have

i) $\hat{R}(X, Y)Z + \hat{R}(Y, Z)X + \hat{R}(Z, X)Y$

$$= (X\alpha)\{\eta(Z)\Phi Y - g(\Phi Y, Z)\xi\} + (X\beta)\{\eta(Y)Z - g(Y, Z)\xi\}$$

$$+ (Y\alpha)\{\eta(Z)\Phi X - g(\Phi X, Z)\xi\} + (Y\beta)\{\eta(Z)X - g(X, Z)\xi\}$$

$$+ \alpha^2 \left\{ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi - g(\Phi Y, Z)\eta(X)\xi \right\} + g(Z, \Phi Y)\Phi X + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X$$

$$+ \beta^2 \{(Y, Z)X - g(Y, Z)Y\}$$

$$+ \alpha\beta \left\{ g(\Phi X, Z)Y - g(\Phi Y, Z)X + g(X, Z)\Phi Y - g(Y, Z)\Phi X \right\} + g(\Phi X, Z)\eta(Y)\xi - g(\Phi Y, Z)\eta(X)\xi + \eta(X)\eta(Z)\Phi Y - \eta(Y)\eta(Z)\Phi X$$

(3.4)

ii) $\hat{R}(X, Y, Z, W) = 0$,

iii) $\hat{R}(X, Y, Z, W) + \hat{R}(Y, X, Z, W) = 0$,

iv) $\hat{R}(X, Y, Z, W) - \hat{R}(Z, W, X, Y)$

$$= (X\alpha)\{\eta(Z)g(\Phi Y, W) - g(\Phi Y, Z)\eta(W)\} - (X\beta)\{\eta(Z)g(Y, W) - g(Y, Z)\eta(W)\}$$

$$+ (Y\alpha)\{\eta(Z)g(\Phi X, W) - g(\Phi X, Z)\eta(W)\} + (Y\beta)\{\eta(Z)g(X, W) - g(X, Z)\eta(W)\}$$

$$+ (Z\alpha)\{\eta(X)g(\Phi W, Y) - g(\Phi W, X)\eta(Y)\} + (Z\beta)\{\eta(X)g(W, Y) - g(W, X)\eta(Y)\}$$

$$+ (W\alpha)\{\eta(X)g(\Phi Z, Y) - g(\Phi Z, X)\eta(Y)\} - (W\beta)\{\eta(X)g(Z, Y) - g(Z, X)\eta(Y)\}$$

$$+ 2\alpha\beta \left\{ g(\Phi X, Z)\{g(Y, W) + \eta(Y)\eta(W)\} + g(X, \Phi W)\{g(Y, Z) + \eta(Y)\eta(Z)\} \right\}$$

$$+ 2\alpha\beta \left\{ g(\Phi Y, Z)\{g(X, W) + \eta(X)\eta(W)\} + g(Y, \Phi W)\{g(X, Z) + \eta(X)\eta(Z)\} \right\}$$

(3.4)

Here, $\hat{R}$ is Riemannian Christoffel curvature tensor of type (0,4) on $M$ with respect to Schouten-Van Kampen connection $\hat{\nabla}$ defined by

$$\hat{R}(X, Y, Z, W) = g(\hat{R}(X, Y)Z, W),$$

for all $X, Y, Z, W \in \chi(M)$.

**Proof.** From first Bianchi identity, (2.3) and (3.4), we get (i).

If $R$ is Riemannian Christoffel tensor field of type (0,4) on $M$ with respect to Levi-Civita connection $\nabla$ defined by $R(X, Y, Z, W) = g(R(X, Y)Z, W)$, then it is well known that

$$\begin{cases}
\hat{R}(X, Y, Z, W) + \hat{R}(X, Y, W, Z) = 0, \\
\hat{R}(X, Y, Z, W) + \hat{R}(Y, X, Z, W) = 0, \\
\hat{R}(X, Y, Z, W) - \hat{R}(Z, W, X, Y) = 0.
\end{cases}$$

(3.5)

So using (2.3), (3.4) and (3.5), we obtain (ii)-(iv).

So, we can give the following Proposition:
Proposition 3. On a trans-Sasakian manifold, we have
\[ \tilde{R}(X,Y)\xi = \tilde{R}(\xi,X)\xi = 0, \] (3.6)
\[ \tilde{R}(\xi,X)Y = 2\alpha\beta \{\eta(Y)\Phi X - g(\Phi X,Y)\xi\} - (\xi\beta) \{\eta(Y)X - g(X,Y)\xi\} \] (3.7)
\[ + (Y\alpha)\Phi X - (Y\beta)\Phi^2 X + g(\Phi Y,X) \operatorname{grad} \alpha - g(\Phi X,\Phi Y) \operatorname{grad} \beta, \]
\[ \eta(\tilde{R}(X,Y)Z) = g(\tilde{R}(X,Y)Z,\xi) = 0, \] (3.8)
for all \(X,Y,Z \in \chi(M)\).

Proof. Using (2.1)-(2.3) and (2.7) in (3.4), we obtain (3.6).

Using (2.1)-(2.3) and (2.7), (3.4), in (3.9), we have (3.7).

And finally, from the Proposition 2-(ii) and (3.6), we have (3.8). \(\square\)

Proposition 4. Let \(M\) be a \((2n+1)\)-dimensional trans-Sasakian manifold of type \((\alpha,\beta)\). Then, the Ricci tensor \(\tilde{S}\) and scalar curvature \(\tilde{\tau}\) of \(M\) with respect to the Schouten-Van Kampen connection \(\tilde{\nabla}\) is
\[ \tilde{S}(X,Y) = S(X,Y) - (2n - 2)\alpha\beta g(\Phi X,Y) + \{(\xi\beta) + 2n\beta^2\} g(X,Y) \] (3.10)
\[ - 2n\alpha^2 \eta(X)\eta(Y) + \{(\Phi X)\alpha + (2n - 1)(X\beta)\} \eta(Y) \]
and
\[ \tilde{\tau} = \tau + 2n \{2(\xi\beta) - \alpha^2 + (2n + 1)\beta^2 \}, \] (3.11)
respectively. Also, the Ricci operator \(\tilde{Q}\) of \(M\) with respect to the Schouten-Van Kampen connection \(\tilde{\nabla}\) is
\[ \tilde{Q}X = QX - (2n - 2)\alpha\beta X + \{(\xi\beta) + 2n\beta^2\} X \] (3.12)
\[ - 2n\alpha^2 \eta(X)\xi + \{(\Phi X)\alpha + (2n - 1)(X\beta)\} \xi, \]
where \(\tilde{S}(X,Y) = g(\tilde{Q}X,Y)\). Here, \(S, Q\) and \(\tau\) are Ricci tensor, Ricci operator and scalar curvature of \(M\) with respect to the Levi-Civita connection \(\nabla\), respectively and \(S(X,Y) = g(QX,Y)\), for all \(X,Y \in \chi(M)\).

Proof. Using (2.1)-(2.3), (2.9) and (3.4) in
\[ \tilde{S}(X,Y) = \sum_{i=1}^{2n+1} g(\tilde{R}(X,e_i)e_i, Y), \]
we get (3.10), where \(\{e_i\}, (i = 1, 2, ..., 2n + 1)\) is a local orthonormal basis of the trans-Sasakian manifold \(M\). Using (3.10) in
\[ \tilde{\tau} = \sum_{i=1}^{2n+1} \tilde{S}(e_i,e_i), \]
we have (3.11). (3.12) is obvious from the definition of Ricci operator and (3.10). \(\square\)
Corollary 1. Let $M$ be a $(2n + 1)$-dimensional trans-Sasakian manifold of type $(\alpha, \beta)$. Then, we have
\begin{align}
\tilde{S}(X, \xi) &= 0, \\
\tilde{S}(\xi, X) &= (2n - 1) \{(\xi \beta)\eta(X) - (X \beta)\} - (\Phi X)\alpha \tag{3.13}
\end{align}
and
\begin{align}
\tilde{Q}\xi &= (2n - 1) \{(\xi \beta)\xi - \text{grad} \beta\} + \Phi(\text{grad} \alpha), \tag{3.14}
\end{align}
for all $X \in \chi(M)$.

Proof. The proof is obvious from (2.1)-(2.3), (2.10), (3.10) and (3.12). □

If the condition of
\begin{align}
(2n - 1) \text{grad} \beta = \Phi(\text{grad} \alpha) \tag{3.16}
\end{align}
satisfies on the trans-Sasakian manifold $M^{2n+1}$, then we have $\xi \beta = 0$. So, under the condition (3.16), the equations (3.7), (3.10)-(3.15) reduce to
\begin{align}
\tilde{R}(\xi, X)Y &= 2\alpha \beta \{\eta(Y)\Phi X - g(\Phi X, Y)\xi\} + (Y \alpha)\Phi X - (Y \beta)\Phi^2 X \\
&\quad + g(\Phi Y, X)\text{grad} \alpha - g(\Phi X, \Phi Y)\text{grad} \beta, \tag{3.17}
\end{align}
\begin{align}
\tilde{S}(X, Y) &= S(X, Y) - (2n - 2)\alpha \beta g(\Phi X, Y) + 2n\beta^2 g(X, Y) - 2n\alpha^2 \eta(X)\eta(Y), \tag{3.18}
\end{align}
\begin{align}
\tilde{\tau} &= \tau + 2n \{\alpha^2 + (2n + 1)\beta^2\}, \tag{3.19}
\end{align}
\begin{align}
\tilde{Q}X &= QX - (2n - 2)\alpha \beta \Phi X + 2n\beta^2 X - 2n\alpha^2 \eta(X)\xi \tag{3.20}
\end{align}
and
\begin{align}
\tilde{S}(X, \xi) = \tilde{S}(\xi, X) = \tilde{Q}\xi &= 0. \tag{3.21}
\end{align}

From (3.18), we can state the following Theorem:

Theorem 1. Let $M$ be a 3-dimensional trans-Sasakian manifold of type $(\alpha, \beta)$. Then, the Ricci tensor $\tilde{S}$ of $M$ with respect to the Schouten-Van Kampen connection $\tilde{\nabla}$ is symmetric if the condition (3.16) satisfies.

Also, under the condition (3.16), in 3-dimensional trans-Sasakian manifold of type $(\alpha, \beta)$ we have
\begin{align}
\tilde{S}(X, Y) &= S(X, Y) + 2(\beta^2 g(X, Y) - \alpha^2 \eta(X)\eta(Y)), \tag{3.22}
\end{align}
\begin{align}
\tilde{Q}X &= QX + 2(\beta^2 X - \alpha^2 \eta(X)\xi) \tag{3.23}
\end{align}
and
\begin{align}
\tilde{\tau} &= \tau + 2 \{3\beta^2 - \alpha^2\}. \tag{3.24}
\end{align}

Here, let us construct an example of a 3-dimensional trans-Sasakian manifold admitting the Schouten-Van Kampen connection.

Example 1. Consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$.

We choose the vector fields $\{e_1, e_2, e_3\}$ as
\begin{align}
e_1 &= -z \left( \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right), \quad e_2 = -z \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \tag{3.25}
\end{align}
which are linearly independent at each point of $M$.

Let $g$ be the Riemannian metric defined by $g(e_i, e_j) = 0$, $i \neq j$, $i, j = 1, 2, 3$ and $g(e_k, e_k) = 1$, $k = 1, 2, 3$. 

**Proof.** The proof is obvious from (2.1)-(2.3), (2.10), (3.10) and (3.12). □
Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z,e_3)$ for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on $M$. Let $\phi$ be the $(1,1)$-tensor field defined by

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.$$ (3.26)

Using the linearity of $\phi$ and $g$, we have $\eta(e_3) = 1$, $\phi^2 Z = -Z + \eta(Z)e_3$ and $g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U)$, for any $U, Z \in \chi(M)$. Thus, for $e_3 = \xi$, $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.

Now, we have

$$[e_1, e_2] = -ye_2 - z^2 e_3, \quad [e_1, e_3] = -\frac{1}{z} e_1, \quad [e_2, e_3] = -\frac{1}{z} e_2.$$ (3.27)

The Levi-Civita connection $\nabla$ of the metric tensor $g$ is given by Koszul’s formula which is defined as

$$2g(\nabla X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Taking $e_3 = \xi$ and using Koszul’s formula, we get the following

$$\nabla_{e_1} e_1 = \frac{1}{z} e_3, \quad \nabla_{e_1} e_2 = -\frac{1}{2} z^2 e_3, \quad \nabla_{e_1} e_3 = -\frac{1}{2} e_1 + \frac{1}{2} z^2 e_2,$$

$$\nabla_{e_2} e_1 = ye_2 + \frac{1}{2} z^2 e_3, \quad \nabla_{e_2} e_2 = -ye_1 + \frac{1}{2} e_3, \quad \nabla_{e_2} e_3 = -\frac{1}{2} z^2 e_1 - \frac{1}{2} e_2,$$ (3.28)

$$\nabla_{e_3} e_1 = \frac{1}{2} z^2 e_2, \quad \nabla_{e_3} e_2 = -\frac{1}{2} z^2 e_1, \quad \nabla_{e_3} e_3 = 0.$$

From the above equalities, it can be easily seen that $(\phi, \xi, \eta, g)$ is a trans-Sasakian structure on $M$. Consequently, $(M, \phi, \xi, \eta, g)$ a trans-Sasakian manifold with $\alpha = -\frac{1}{2} z^2 \neq 0$ and $\beta = \frac{1}{z}$.

Using the above relations in (3.13), we obtain

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_1} e_3 = 0,$$

$$\nabla_{e_2} e_1 = ye_2, \quad \nabla_{e_2} e_2 = -ye_1, \quad \nabla_{e_2} e_3 = 0,$$ (3.29)

$$\nabla_{e_3} e_1 = \frac{1}{2} z^2 e_2, \quad \nabla_{e_3} e_2 = -\frac{1}{2} z^2 e_1, \quad \nabla_{e_3} e_3 = 0.$$

From the above equalities, we can obtain the components of the curvature tensor with respect to the Levi-Civita connection are as follows:

$$R(e_1, e_2) e_1 = \left( y^2 + \frac{1}{z^2} + \frac{3}{4} z^4 \right) e_2 - yz^2 e_3,$$

$$R(e_1, e_2) e_2 = -\left( y^2 + \frac{1}{z^2} + \frac{3}{4} z^4 \right) e_1 + \frac{y}{z} e_3, \quad R(e_1, e_2) e_3 = yz^2 e_1 - \frac{y}{z} e_2,$$

$$R(e_1, e_3) e_1 = -yz^2 e_2 + \left( \frac{2}{z^2} - \frac{1}{4} z^4 \right) e_3, \quad R(e_1, e_3) e_2 = yz^2 e_1,$$ (3.30)

$$R(e_1, e_3) e_3 = \left( \frac{1}{4} z^4 - \frac{1}{2} z^2 \right) e_1, \quad R(e_2, e_3) e_1 = \frac{y}{z} e_2,$$

$$R(e_2, e_3) e_2 = -\frac{y}{z} e_1 + \left( \frac{2}{z^2} - \frac{1}{4} z^4 \right) e_3, \quad R(e_2, e_3) e_3 = \left( \frac{1}{4} z^4 - \frac{1}{2} z^2 \right) e_2.$$
On the other hand, the components of the curvature tensor with respect to the Schouten-Van Kampen connection are as follows:

\[ \tilde{R}(e_1, e_2)e_1 = (y^2 + \frac{1}{2}z^4) e_2, \quad \tilde{R}(e_1, e_2)e_2 = -(y^2 + \frac{1}{2}z^4) e_1, \quad \tilde{R}(e_1, e_2)e_3 = 0, \]
\[ \tilde{R}(e_1, e_3)e_1 = -yz^2 e_2, \quad \tilde{R}(e_1, e_3)e_2 = yz^2 e_1, \quad \tilde{R}(e_1, e_3)e_3 = 0, \quad (3.31) \]
\[ \tilde{R}(e_2, e_3)e_1 = \frac{y}{z} e_2, \quad \tilde{R}(e_2, e_3)e_2 = -\frac{y}{z} e_1, \quad \tilde{R}(e_2, e_3)e_3 = 0. \]

With the help of the equations (3.30) and (3.31), we get the Ricci tensors of Levi-Civita connection and Schouten-Van Kampen connection, respectively, as follows:

\[ S(e_1, e_1) = -\left(y^2 + \frac{3}{z^2} + \frac{1}{2}z^4\right), \quad S(e_1, e_2) = 0, \quad S(e_1, e_3) = \frac{y}{z}, \]
\[ S(e_2, e_1) = 0, \quad S(e_2, e_2) = -\left(y^2 + \frac{3}{z^2} + \frac{1}{2}z^4\right), \quad S(e_2, e_3) = yz^2, \quad (3.32) \]
\[ S(e_3, e_1) = \frac{y}{z}, \quad S(e_3, e_2) = yz^2, \quad S(e_3, e_3) = \left(\frac{1}{2}z^4 - \frac{4}{z^2}\right) \]

and

\[ \tilde{S}(e_1, e_1) = -\left(y^2 + \frac{1}{2}z^4\right), \quad \tilde{S}(e_1, e_2) = \tilde{S}(e_1, e_3) = 0, \]
\[ \tilde{S}(e_2, e_1) = 0, \quad \tilde{S}(e_2, e_2) = -\left(y^2 + \frac{1}{2}z^4\right), \quad \tilde{S}(e_2, e_3) = 0, \quad (3.33) \]
\[ \tilde{S}(e_3, e_1) = \frac{y}{z}, \quad \tilde{S}(e_3, e_2) = yz^2, \quad \tilde{S}(e_3, e_3) = 0. \]

Therefore, the scalar curvatures of Levi-Civita connection and Schouten-Van Kampen connection are obtained as

\[ \tau = \sum_{i=1}^{3} S(e_i, e_i) = -2\left(\frac{1}{4}z^4 + \frac{5}{z^2} + y^2\right) \quad (3.34) \]

and

\[ \tilde{\tau} = \sum_{i=1}^{3} \tilde{S}(e_i, e_i) = -(2y^2 + z^4), \quad (3.35) \]

respectively.

4. \( \eta \)-Ricci Solitons on 3-Dimensional Trans-Sasakian Manifolds with Schouten-Van Kampen Connection

Throughout this section, we assume that the condition (3.16) is satisfied on 3-dimensional trans-Sasakian manifold.

Let \((M, \Phi, \eta, \xi, g)\) be a 3-dimensional trans-Sasakian manifold. The data \((g, V, \lambda, \mu)\) which satisfy the equation

\[ L_V g + 2S + 2\lambda g + 2\mu \eta \otimes \eta = 0 \quad (4.1) \]

is called an \( \eta \)-Ricci soliton on \( M \) [7]. Here, \( L_V \) is the Lie derivative operator along the vector field \( V \), \( S \) is the Ricci curvature tensor field with respect to Levi-Civita connection and \( \lambda \) and \( \mu \) are real constants. Especially, if \( \mu = 0 \), then \((g, V, \lambda)\) is a Ricci soliton and it is called shrinking, steady or expanding according to \( \lambda \) is negative, zero or positive, respectively [8].
Let \((g, V, \lambda, \mu)\) be an \(\eta\)-Ricci soliton on a trans-Sasakian manifold with respect to Schouten-Van Kampen connection. Then we have
\[
(\tilde{L}_V)(Y, Z) + 2\tilde{S}(Y, Z) + 2\lambda g(Y, Z) + 2\mu \eta(Y) \eta(Z) = 0,
\]
where \(\tilde{L}_V\) is the Lie derivative along the vector field \(V\) on \(M\) and \(\tilde{S}\) is the Ricci curvature tensor field with respect to the Schouten-Van Kampen connection \(\tilde{\nabla}\).

From (2.13), we have
\[
(\tilde{L}_V)(Y, Z) = g(\tilde{\nabla}_Y V, Z) + g(Y, \tilde{\nabla}_Z V)
\]
\[
= (L_V)(Y, Z) - \alpha g(V, \eta(Z) \Phi Y + \eta(Y) \Phi Z)
+ \beta g(\eta(Z) Y + \eta(Y) Z - 2g(Y, Z) \xi, V).\]

So, using (3.22) and (4.3) in (4.2), we obtain that
\[
(L_V)(Y, Z) = 2S(Y, Z) + 2\lambda g(Y, Z) + 2\mu \eta(Y) \eta(Z)
- \alpha \{g(V, \eta(Z) \Phi Y + \eta(Y) \Phi Z) + 4\eta(Y) \eta(Z)\}
+ \beta \{g(\eta(Z) Y + \eta(Y) Z - 2g(Y, Z) \xi, V) + 4\beta g(Y, Z)\} = 0.
\]

Hence, we can state the following Theorem:

**Theorem 2.** An \(\eta\)-Ricci soliton \((g, V, \lambda, \mu)\) on a trans-Sasakian manifold is invariant under Schouten-Van Kampen connection if and only if the relation
\[
\alpha \{g(V, \eta(Z) \Phi Y + \eta(Y) \Phi Z) + 4\eta(Y) \eta(Z)\}
= \beta \{g(\eta(Z) Y + \eta(Y) Z - 2g(Y, Z) \xi, V) + 4\beta g(Y, Z)\}
\]
holds for all vector fields \(Y, Z\) and \(V\).

Now, let \(V\) be pointwise collinear with \(\xi\), i.e. \(V = b\xi\), where \(b\) is a function on the 3-dimensional trans-Sasakian manifold. Then \((\tilde{L}_V)g + 2\tilde{S} + 2\lambda g + 2\mu \eta \otimes \eta)(Y, Z) = 0\) implies
\[
g(\tilde{\nabla}_Y b\xi, Z) + g(Y, \tilde{\nabla}_Z b\xi) + 2\tilde{S}(Y, Z) + 2\lambda g(Y, Z) + 2\mu \eta(Y) \eta(Z) = 0\]
and using (2.14) and (3.22) in (4.6), we have
\[
(Yb)\eta(Z) + (Zb)\eta(Y) + 2S(Y, Z) + 4\beta^2 g(Y, Z)
- 4\alpha^2 \eta(Y) \eta(Z) + 2\lambda g(Y, Z) + 2\mu \eta(Y) \eta(Z) = 0.
\]
Taking \(Y = \xi\) in (4.7) and using (2.10), we get
\[
(\xi b)\eta(Z) + (Zb) + 2(\lambda + \mu) \eta(Z) = 0.
\]
Again taking \(Z = \xi\) in (4.8), we have \(\xi b = -(\lambda + \mu)\) and using this expression in (4.8) we get
\[
db = -(\lambda + \mu) \eta,
\]
where \(d\) is the differential operator. Applying \(d\) to (4.9), we have \((\lambda + \mu) d\eta = 0\). Since \(d\eta \neq 0\), we get \(\lambda + \mu = 0\). Using this equation in (4.9), we obtain that \(b\) is a constant.

So, from (4.7) we have
\[
S(Y, Z) = -(2\beta^2 + \lambda) g(Y, Z) + (2\alpha^2 - \mu) \eta(Y) \eta(Z).
\]

Thus, we have
Theorem 3. If \((g, V, \lambda, \mu)\) is an \(\eta\)-Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection and \(V\) is pointwise collinear with \(\xi\), then \(V\) is a constant multiple of \(\xi\) and \(M\) is an \(\eta\)-Einstein manifold with respect to Levi-Civita connection.

Also, from (4.10) we can state the following Corollary:

Corollary 2. If \((g, V, \lambda, \mu)\) is an \(\eta\)-Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection and \(V\) is pointwise collinear with \(\xi\), then \(V\) is a constant multiple of \(\xi\) and \(M\) is of constant scalar curvature provided \(\alpha\) and \(\beta\) are constants.

Here, let \((g, \xi, \lambda, \mu)\) be an \(\eta\)-Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection. Taking \(V = \xi\) in (4.12), then we have

\[
(\tilde{L}_\xi g)(Y, Z) + 2\tilde{S}(Y, Z) + 2\lambda g(Y, Z) + 2\mu \eta(Y)\eta(Z) = 0. \tag{4.11}
\]

Using (2.14) and (3.22) in (4.11), we have

\[
S(Y, Z) = -(2\beta^2 + \lambda)g(Y, Z) + (2\alpha^2 - \mu)\eta(Y)\eta(Z). \tag{4.12}
\]

So, we have

Theorem 4. If \((g, V, \lambda, \mu)\) is an \(\eta\)-Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection, then \(M\) is an \(\eta\)-Einstein manifold with respect to Levi-Civita connection.

Taking \(Z = \xi\) in (4.12) and using (2.10), we get

\[
\lambda + \mu = 0. \tag{4.13}
\]

Hence,

Theorem 5. If \((g, V, \lambda, \mu)\) is an \(\eta\)-Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection, then \(\lambda = -\mu\).

Finally, let us give some results for Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection. If \((g, V, \lambda)\) is a Ricci soliton on a trans-Sasakian manifold with respect to Schouten-Van Kampen connection, then taking \(\mu = 0\) in (4.2) we have

\[
(\tilde{L}_V g)(Y, Z) + 2\tilde{S}(Y, Z) + 2\lambda g(Y, Z) = 0. \tag{4.14}
\]

Using (3.22) and (4.3) in (4.14), we have

\[
(L_V g)(Y, Z) = 2S(Y, Z) + 2\lambda g(Y, Z)
- \alpha \{g(V, \eta(Z)\Phi Y + \eta(Y)\Phi Z) + 4\alpha \eta(Y)\eta(Z)\}
+ \beta \{g(\eta(Z)Y + \eta(Y)Z - 2g(Y, Z)\xi, V) + 4\beta g(Y, Z)\} = 0. \tag{4.15}
\]

So, we can state the following Theorem:

Theorem 6. A Ricci soliton \((g, V, \lambda)\) on a trans-Sasakian manifold is invariant under Schouten-Van Kampen connection if and only if the relation (4.3) holds for all vector fields \(Y, Z\) and \(V\).
If $V$ is pointwise collinear with $\xi$, then taking $\mu = 0$ in (4.7)-(4.9) we have $b$ is constant and the equation (4.10) reduces to
\[
S(Y, Z) = -2\beta^2 g(Y, Z) + 2\alpha^2 \eta(Y)\eta(Z).
\] (4.16)
So, we have

**Theorem 7.** If $(g, V, \lambda)$ is a Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection and $V$ is pointwise collinear with $\xi$, then $V$ is a constant multiple of $\xi$ and $M$ is an $\eta$-Einstein manifold with respect to Levi-Civita connection.

**Corollary 3.** If $(g, V, \lambda)$ is a Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection and $V$ is pointwise collinear with $\xi$, then the scalar curvature of $M$ with respect to Levi-Civita connection is $\tau = 2(\alpha^2 - 3\beta^2)$. Also, using this and (4.16) in (3.22)-(3.24), we have $\tilde{S} = \tilde{Q} = \tilde{\tau} = 0$.

From (4.16), we can state the following Corollary:

**Corollary 4.** If $(g, V, \lambda)$ is a Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection and $V$ is pointwise collinear with $\xi$, then $V$ is a constant multiple of $\xi$ and $M$ is of constant scalar curvature provided $\alpha$ and $\beta$ are constants.

Now, if $(g, \xi, \lambda)$ is a Ricci soliton on a trans-Sasakian manifold with respect to Schouten-Van Kampen connection, then taking $\mu = 0$ in (4.11), the equation (4.12) reduces to
\[
S(Y, Z) = -(2\beta^2 + \lambda) g(Y, Z) + 2\alpha^2 \eta(Y)\eta(Z).
\] (4.17)
Thus, we get

**Theorem 8.** If $(g, \xi, \lambda)$ is a Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to Schouten-Van Kampen connection, then $M$ is an $\eta$-Einstein manifold with respect to Levi-Civita connection.

Taking $Z = \xi$ in (4.17) and using (2.10), we have $\lambda = 0$. Hence,

**Theorem 9.** A Ricci soliton $(g, \xi, \lambda)$ on a trans-Sasakian manifold with respect to Schouten-Van Kampen connection is always steady.

**References**


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