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ON GENERALIZED STATISTICAL CONVERGENCE IN g-METRIC SPACES

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Abstract. This manuscript focuses on the investigation of λ-statistical convergence, λ-statistically Cauchy sequences in g-metric spaces, and the relationship between these concepts. We investigate almost λ-statistical convergence by using the notion of (V, λ)-summability to generalize the concept of statistical convergence in g-metric space. Moreover, we expand the definition of λ-statistical convergence to encompass invariant statistical convergence as well as invariant λ-statistical convergence in g-metric spaces. We delve into the examination of their intriguing and fundamental properties.

1. Introduction

For numerous decades, the study of summability theory and sequence convergence has been one of the most significant and active areas of academic effort in pure mathematics. Its substantial works may also be used in topology, functional analysis, Fourier analysis, measure theory, applied mathematics, mathematical modeling, computer science, and other fields. In recent years, many mathematicians have used the concept of statistical convergence of sequences, which was first introduced by Fast [8] as an extension of the usual concept of sequential limits, as a tool to solve many open problems in the area of sequence spaces and summability theory, as well as in some other applications. One may refer to [6, 10]. Mursaleen [15], on the other hand, introduced the concept of λ-statistical convergence as a novel approach and explored its connections to statistical convergence, strongly Cesro summability, and strongly (V, λ)-summability. In recent years, Braha [3, 4], Esi et al. [7], Hazarika et al. [11], Kii and Nuray [13], Sava [17], and Sava and Nuray [18] have generalized the notions of asymptotically equivalent, λ-statistical convergence, almost λ-statistical convergence, and invariant statistical convergence. For further background on sequence spaces and related topics, readers are encouraged to refer to the monographs [2] and [16].

Various methods exist for extending the notion of a distance function (refer to [12] for details). One noteworthy approach is the concept of a G-metric space,

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introduced by Mustafa and Sims [14], which presents a fresh and distinctive generalization of the ordinary metric. In this framework, metrics represent the distance between three locations. Choi et al. proposed a concept called g-metric, which extends the idea of a distance function, in [5]. The g-metric with degree n is a distance function involving $n+1$ points, and it provides a generalization of both the ordinary distance between two points and the G-metric between three points. Abazari recently introduced the notion of statistical g-convergence in [1], extending the concept of a metric.

The primary objective of this research is to introduce the concepts of λ-statistically convergent sequences and λ-statistically g-Cauchy sequences, and explore their properties in g-metric spaces. Additionally, we extend the definition of λ-statistical convergence to invariant statistical convergence and invariant λ-statistical convergence, and examine their relationship with $g\left[\hat{V}_\lambda\right]$ and $g\hat{S}_\lambda$. Furthermore, we will present natural inclusion theorems in addition to these definitions.

2. Preliminaries

In this section, we will review specific definitions and results that form the basis of the current study. We will begin by presenting several definitions.

The main concept underlying statistical convergence is the notion of natural density. The natural density of a set $A \subseteq \mathbb{N}$ is denoted and defined as follows:

$$\delta(A) = \lim_n \tfrac{1}{n} \left|\{k \in A : k \leq n\}\right|,$$

where the vertical bars denote the cardinality of the set enclosed.
A real-valued sequence $x=(x_k)$ is said to be statistically convergent to the real number x if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\}) = 0.$$

We shall also use S to denote the set of all statistically convergent sequences.

The concept of λ-statistical convergence of sequences $x=(x_k)$ of real numbers has been studied by Mursaleen [15]. Let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive real numbers that tends to infinity, satisfying $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$. The generalized de la Valle-Poussin mean is defined as

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$$

where $I_n = [n-\lambda_n+1, n]$ for $n = 1, 2, ...$
If $\lambda_n = n$, then (V,λ)-summability reduces to $(C,1)$-summability. We denote

$$[C,1] = \left\{ x=(x_n) : \exists L \in \mathbb{R},\ \lim_n n^{-1} \sum_{k=1}^{n} d(x_k, x) = 0 \right\}$$

and

$$[V,\lambda] = \left\{ x=(x_n) : \exists L \in \mathbb{R},\ \lim_n \lambda_n^{-1} \sum_{k \in I_n} d(x_k, x) = 0 \right\}$$

for the sets of sequences $x=(x_k)$ that are strongly Cesro summable and strongly (V,λ)-summable to a number x, respectively.

A sequence of real numbers $x = (x_k)$ is said to be λ-statistically convergent to the number x if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : d(x_k, x) \geq \varepsilon\}| = 0.$$

In this case, we denote the λ-statistical limit of (x_k) as S_λ-$\lim x_k = x$ or $x_k \to x(S_\lambda)$.

Remark. *If $\lambda_n = n$, then S_λ is equivalent to S.*

Throughout the paper, let (Y, g) denote a g-metric space, and let (x_n) be a sequence in Y.

Definition 2.1 ([14])**.** *Let Y be a nonempty set, and a function $G : Y \times Y \times Y \to \mathbb{R}^+$ that satisfies the following five properties is called a generalized metric or, briefly, a G-metric on Y. The pair (Y, G) is referred to as a G-metric space.*
i) $G(u, v, x) = 0$ if $u = v = x$,
ii) $0 < G(u, u, v)$; for each $u, v \in G$, with $u \neq v$,
iii) $G(u, u, v) \leq G(u, v, x)$, for each $u, v, x \in Y$ with $x \neq v$,
iv) $G(u, v, x) = G(u, x, v) = G(v, x, u) = \cdots$ (symmetry in all three variables),
v) $G(u, v, x) \leq G(u, \alpha, \alpha) + G(\alpha, v, x)$, for each $u, v, x, \alpha \in Y$ (rectangle inequality).

Subsequently, Choi et al. [5] introduced g-metric functions of degree n.

Definition 2.2. *Let Y be a nonempty set. A function $g : Y^{j+1} \to \mathbb{R}^+$ that satisfies the following features is called a g-metric with order j on Y. The pair (Y, g) is referred to as a g-metric space.*
gi) $g(x_0, x_1, ..., x_j) = 0$ iff $x_0 = x_1 = ... = x_j$,
gii) $g(x_0, x_1, ..., x_j) = g\left(x_{\rho(0)}, x_{\rho(1)}, ..., x_{\rho(j)}\right)$, for permutation ρ on $\{0, 1, ..., j\}$,
giii) $g(x_0, x_1, ..., x_j) \leq g(q_0, q_1, ..., q_j)$, for each $(x_0, x_1, ..., x_j)$, $(q_0, q_1, ..., q_j) \in Y^{j+1}$ with

$$\{x_i : i = 0, 1, ..., j\} \subseteq \{q_i : i = 0, 1, ..., j\},$$

giv) For all $x_0, x_1, ..., x_s, q_0, q_1, ..., q_t, v \in Y$ with $s + t + 1 = j$,

$$g(x_0, x_1, ..., x_s, q_0, q_1, ..., q_t) \leq g(x_0, x_1, ..., x_s, v, v, ..., v) + g(q_0, q_1, ..., q_t, v, v, ..., v).$$

It is obvious that when $j = 1$, we have an ordinary metric space, and when $j = 2$, we have a G-metric space.

The following theorem will be required in the main findings.

Theorem 2.3. *Let Y be a nonempty set, and let g be a metric with order j on Y. In this context, the following properties are provided:*

1) $g(\underbrace{x, ..., x}_{s \text{ times}}, q, ..., q) \leq g(\underbrace{x, ..., x}_{s \text{ times}}, u, ..., u) + g(\underbrace{u, ..., u}_{s \text{ times}}, q, ..., q)$,

2) $g(x, q, ..., q) \leq g(x, u, ..., u) + g(u, q, ..., q)$,

3) $g(\underbrace{x, ..., x}_{s \text{ times}}, u, ..., u) \leq s g(x, u, ..., u)$ *and* $g(\underbrace{x, ..., x}_{s \text{ times}}, u, ..., u) \leq (j + 1 - s) g(u, x, ..., x)$,

4) $g(x_0, x_1, ..., x_j) \leq \sum_{i=0}^{n} g(x_i, u, ..., u)$,

5) $|g(q, x_1, x_2, ..., x_j) - g(u, x_1, x_2, ..., x_j)| \leq \max\{g(q, u, ..., u), g(u, q, ..., q)\}$, *6)* $\left| g(\underbrace{x,...,x}_{s\ times},u,...,u) - g(\underbrace{x,...,x}_{s'\ times},u,...,u) \right| \leq |s-s'|\, g(x,u,...,u)$,

7) $g(x,u,...,u) \leq (1+(s-1))(j+1-s)\, g(\underbrace{x,...,x}_{s\ times},u,...,u)$.

Definition 2.4 ([1])**.** *Let (x_n) be a sequence in a g-metric space (Y,g).*

(i) The sequence (x_n) is said to be statistically convergent to x, provided that for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{j!}{n^j} \left| \left\{ i_1, i_2, ..., i_j \leq n : g\left(x, x_{i_1}, x_{i_2}, ..., x_{i_j}\right) \geq \varepsilon \right\} \right| = 0,$$

and is denoted by $gS\text{-}\lim_{n\to\infty} x_n = x$.

(ii) The sequence (x_n) is called statistically g-Cauchy, provided that for all $\varepsilon > 0$, there exists $i_\varepsilon \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{j!}{n^j} \left| \left\{ i_1, i_2, ..., i_j \leq n : g\left(x_{i_\varepsilon}, x_{i_1}, x_{i_2}, ..., x_{i_j}\right) \geq \varepsilon \right\} \right| = 0.$$

3. Main Results

Based on the aforementioned definitions and results, we aim to introduce novel concepts of λ-statistically convergent sequences in the context of metrics on g-metric spaces in this section. Furthermore, we will provide natural inclusion theorems in addition to these definitions.

Now, we are prepared to define λ-statistical convergence in the g-metric space (X,g).

Definition 3.1. *A sequence $x = (x_n)$ in a g-metric space (X,g) is said to be λ-statistically convergent to x if for every $\varepsilon > 0$,*

$$\delta_\lambda^j (A(n)) = \lim_{n\to\infty} \frac{j!}{(\lambda_n)^j} \left| \left\{ i_1, i_2, ..., i_j \in I_n : g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \geq \varepsilon \right\} \right| = 0,$$

or

$$\lim_{n\to\infty} \frac{j!}{(\lambda_n)^j} \left| \left\{ i_1, i_2, ..., i_j \in I_n : g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) < \varepsilon \right\} \right| = 1.$$

In that case, we denote $gS_\lambda\text{-}\lim x_n = x$ or $x_n \to x\,(gS_\lambda)$. When $\lambda_n = n$ for all n, the notion of gS_λ-statistical convergence for sequences reduces to the concept of g-statistical convergence as defined in [1, Definition 2.4(i)].

Theorem 3.2. *Every convergent sequence in a g-metric space is also λ-statistically convergent.*

Proof. According to the definition provided in [5, Definition 4.1], let us assume that the sequence (x_n) g-converges to x. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$i_1, i_2, ..., i_j > N \Longrightarrow g\left(x, x_{i_1}, x_{i_2}, ..., x_{i_j}\right) < \varepsilon.$$

Let us consider

$$A(n) = \left\{ i_1, i_2, ..., i_j \in I_n : g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) < \varepsilon \right\}.$$

We can observe that

$$|A(n)| \geq \binom{\lambda_n - \lambda_N}{j},$$

and

$$\lim_{n\to\infty}\frac{j!}{(\lambda_n)^j}|A(n)|\geq\lim_{n\to\infty}\frac{j!}{(\lambda_n)^j}\begin{pmatrix}\lambda_n-\lambda_N\\ j\end{pmatrix}=1.$$

Therefore, we conclude that gS_λ-$\lim x_n=x$ as desired. □

The following theorem establishes the uniqueness of the statistical limit in a g-metric space.

Theorem 3.3. *If (x_n) is a sequence in a g-metric space (X,g) such that gS_λ-$\lim x_n=x$ and gS_λ-$\lim x_n=y$, then $x=y$.*

Proof. For any arbitrary $\varepsilon>0$, we define the sets:

$$A(n):=\left\{i_1,i_2,...,i_j\in I_n:g\left(x,x_{i_1},x_{i_2},...,x_{i_j}\right)\geq\frac{\varepsilon}{2j}\right\},$$

$$B(n):=\left\{i_1,i_2,...,i_j\in I_n:g\left(y,x_{i_1},x_{i_2},...,x_{i_j}\right)\geq\frac{\varepsilon}{2j}\right\}.$$

Since gS_λ-$\lim x_n=x$ and gS_λ-$\lim x_n=y$, we have $\delta_\lambda^j\left(A(n)\right)=0$ and $\delta_\lambda^j\left(B(n)\right)=0$.

Let $C(n):=A(n)\cup B(n)$. Then $\delta_\lambda^j\left(C(n)\right)=0$, which implies $\delta_\lambda^j\left(C^c(n)\right)=1$.

Suppose $i_1,i_2,...,i_j\in C^c(n)$. By Theorem 2.3, we have

$$\begin{aligned}g(x,y,y,...,y)&\leq g\left(x,x_{i_1},x_{i_1},...,x_{i_1}\right)+g\left(x_{i_1},y,y,...,y\right)\\&\leq g\left(x,x_{i_1},x_{i_1},...,x_{i_1}\right)+j\left(g\left(y,x_{i_1},x_{i_1},...,x_{i_1}\right)\right)\\&\leq g\left(x,x_{i_1},x_{i_2},...,x_{i_j}\right)+j\left(g\left(y,x_{i_1},x_{i_2},...,x_{i_j}\right)\right)\\&\leq j\left(g\left(x,x_{i_1},x_{i_2},...,x_{i_j}\right)+g\left(y,x_{i_1},x_{i_2},...,x_{i_j}\right)\right)\\&<j\left(\frac{\varepsilon}{2j}+\frac{\varepsilon}{2j}\right)=\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we have

$$g(x,y,y,...,y)=0,$$

which implies $x=y$. □

Definition 3.4. *A sequence $x=(x_n)$ in a g-metric space (X,g) is said to be λ-statistically g-Cauchy if for each $\varepsilon>0$, there exists $i_0\in I_n$ such that*

$$\lim_{n\to\infty}\frac{j!}{(\lambda_n)^j}\left|\left\{i_1,i_2,...,i_j\in I_n:g\left(x_{i_0,}x_{i_1},x_{i_2},...,x_{i_j}\right)\right\}\geq\varepsilon\right|=0.$$

If $\lambda_n=n$ for all n, the notion of λ-statistically g-Cauchy sequence is equivalent to the concept of statistical g-Cauchy sequence as defined in [1] (Definition 2.4(ii)).

Theorem 3.5. *Let (X,g) be g-metric space. If the sequence (x_n) is λ-statistically convergent, then (x_n) is λ-statistically g-Cauchy.*

Proof. Let (x_n) be a λ-statistically convergent sequence in g-metric space (X,g) and $\varepsilon>0$, then

$$\lim_{n\to\infty}\frac{j!}{(\lambda_n)^j}\left|\left\{i_1,i_2,...,i_j\in I_n:g\left(x,x_{i_1},x_{i_2},...,x_{i_j}\right)<\frac{\varepsilon}{j\,(j+1)}\right\}\right|=1.$$

By the monotonicity condition of the g-metric and Theorem 2.3, we can conclude that

$$\begin{aligned} g\left(x_{i_0},x_{i_1},x_{i_2},...,x_{i_j}\right) &\leq \sum_{k=0}^{j} g\left(x_{i_k},x,x,...,x\right) \\ &\leq \sum_{k=0}^{j} jg\left(x,x_{i_k},x_{i_k},...,x_{i_k}\right) \\ &< j\left(j+1\right) g\left(x,x_{i_1},x_{i_2},...,x_{i_j}\right) \\ &< j\left(j+1\right)\frac{\varepsilon}{j\left(j+1\right)} = \varepsilon. \end{aligned}$$

From the above inequality, we have

$$\begin{aligned} &\left\{i_1,i_2,...,i_j \in I_n : g\left(x,x_{i_1},x_{i_2},...,x_{i_j}\right) < \frac{\varepsilon}{j\left(j+1\right)}\right\} \\ &\subseteq \left\{i_1,i_2,...,i_j \in I_n : g\left(x_{i_0},x_{i_1},x_{i_2},...,x_{i_j}\right) < \varepsilon\right\}. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{j!}{\left(\lambda_n\right)^j} \left|\left\{i_1,i_2,...,i_j \in I_n : g\left(x_{i_0},x_{i_1},x_{i_2},...,x_{i_j}\right) < \varepsilon\right\}\right| = 1$$

is obtained and it is demonstrated that (x_n) is a λ-statistically g-Cauchy sequence in (X,g). □

Let Λ be the set of all non-decreasing sequences $\lambda = (\lambda_n)$ of positive numbers that tend to infinity and satisfy $\lambda_{n+1} \leq \lambda_n + 1$ with $\lambda_1 = 1$. Additionally, we denote

$$g\left[C,1\right] = \left\{x = (x_j) : \exists x \in \mathbb{R},\ \lim_n \frac{j!}{n^j} \sum_{i_1,i_2,...,i_j=1}^{n} g\left(x_{i_1},x_{i_2},...,x_{i_j},x\right) = 0\right\}$$

and

$$g\left[V,\lambda\right] = \left\{x = (x_j) : \exists x \in \mathbb{R},\ \lim_n \frac{j!}{\left(\lambda_n\right)^j} \sum_{i_1,i_2,...,i_j \in I_n} g\left(x_{i_1},x_{i_2},...,x_{i_j},x\right) = 0\right\}$$

for the sets of sequences $x = (x_j)$ which are strongly g-Cesáro summable and strongly $g\left(V,\lambda\right)$-summable to a number x, i.e. $x_n \to x\left(g\left[C,1\right]\right)$ and $x_n \to x\left(g\left[V,\lambda\right]\right)$ respectively.

Proofs of the following result are straightforward and omitted.

Theorem 3.6. *Let (x_n) be a sequence in g-metric space (X,g). Then*
i) If $x_n \to x\left(g\left[C,1\right]\right)$, then $x_n \to x\left(gS_\lambda\right)$.
ii) If (X,g) is bounded and $x_n \to x\left(gS_\lambda\right)$, then $x_n \to x\left(g\left[C,1\right]\right)$.

Theorem 3.7. *Let (X,g) be a g-metric space. Then, following statements hold:*
(i) If $x_n \to x\left(g\left[V,\lambda\right]\right)$ then $x_n \to x\left(gS_\lambda\right)$, and the inclusion $g\left[V,\lambda\right] \subseteq gS_\lambda$ is proper.
(ii) If $x \in \ell_\infty$ and $x_n \to x\left(gS_\lambda\right)$, then $x_n \to x\left(g\left[V,\lambda\right]\right)$.
(iii) $gS_\lambda \cap \ell_\infty = g\left[V,\lambda\right] \cap \ell_\infty$.

Proof. (i) Let $x_n \to x\,(g\,[V,\lambda])$. Then, for $\varepsilon > 0$, we have

$$\sum_{i_1,i_2,...,i_j \in I_n} g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \geq \sum_{\substack{i_1,i_2,...,i_j \in I_n \\ g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \geq \varepsilon}} g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right)$$
$$\geq \varepsilon \left|\left\{i_1, i_2, ..., i_j \in I_n : g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \geq \varepsilon\right\}\right|.$$

Hence, we have $x_n \to x\,(gS_\lambda)$.

It is easy to see that the inclusion $g\,[V,\lambda] \subseteq gS_\lambda$ is proper.

(ii) To prove part (ii), we assume that $x = (x_n)$ is in ℓ_∞ and $x_n \to x\,(gS_\lambda)$. Then, we can assume that

$$g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \leq M \text{ for all } j.$$

Given $\varepsilon > 0$, we obtain

$$\frac{j!}{(\lambda_n)^j} \sum_{i_1,i_2,...,i_j \in I_n} g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right)$$
$$= \frac{j!}{(\lambda_n)^j} \sum_{\substack{i_1,i_2,...,i_j \in I_n \\ g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \geq \varepsilon}} g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right)$$
$$+ \frac{j!}{(\lambda_n)^j} \sum_{\substack{i_1,i_2,...,i_j \in I_n \\ g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) < \varepsilon}} g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right)$$
$$\leq M \frac{j!}{(\lambda_n)^j} \left|\left\{i_1, i_2, ..., i_j \in I_n : g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}.$$

As a result, we can conclude that $x_n \to x\,(g\,[V,\lambda])$. Moreover, we can express this as

$$\frac{j!}{n^j} \sum_{i_1,i_2,...,i_j=1}^{n} g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) = \frac{j!}{n^j} \sum_{i_1,i_2,...,i_j=1}^{n-\lambda_n} g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right)$$
$$+ \frac{j!}{n^j} \sum_{i_1,i_2,...,i_j \in I_n} g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right)$$
$$\leq \frac{j!}{(\lambda_n)^j} \sum_{i_1,i_2,...,i_j=1}^{n-\lambda_n} g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right)$$
$$+ \frac{j!}{(\lambda_n)^j} \sum_{i_1,i_2,...,i_j \in I_n} g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right)$$
$$\leq \frac{2j!}{(\lambda_n)^j} \sum_{i_1,i_2,...,i_j \in I_n} g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right).$$

Since $x_n \to x\,(g\,[V,\lambda])$, the desired $x_n \to x\,(g\,[C,1])$ is attained.

(iii) This follows directly from (i) and (ii). □

It is evident that $x_n \to x\,(gS_\lambda) \subseteq x_n \to x\,(gS)$ for all λ, since $\frac{(\lambda_n)^j}{n^j}$ is bounded by 1. Therefore, we establish the following relation.

Theorem 3.8. $gS \subseteq gS_\lambda$ *iff* $\liminf \frac{(\lambda_n)^j}{n^j} > 0$.

Proof. For a given $\varepsilon > 0$, we observe that

$$\begin{aligned} &\left\{i_1, i_2, ..., i_j \in I_n : g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \geq \varepsilon\right\} \\ &\subset \left\{i_1, i_2, ..., i_j \leq n : g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \geq \varepsilon\right\}. \end{aligned}$$

This yields

$$\begin{aligned} &\frac{j!}{n^j}\left|\left\{i_1, i_2, ..., i_j \leq n : g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \geq \varepsilon\right\}\right| \\ &\geq \frac{j!}{n^j}\left|\left\{i_1, i_2, ..., i_j \in I_n : g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \geq \varepsilon\right\}\right| \\ &\geq \frac{(\lambda_n)^j}{n^j}.\frac{j!}{(\lambda_n)^j}\left|\left\{i_1, i_2, ..., i_j \in I_n : g\left(x_{i_1}, x_{i_2}, ..., x_{i_j}, x\right) \geq \varepsilon\right\}\right|. \end{aligned}$$

Taking the limits as $n \to \infty$ and employing the fact that $\liminf \frac{(\lambda_n)^j}{n^j} > 0$, we obtain

$$x_j \to x\,(gS) \Rightarrow x_j \to x\,(gS_\lambda)\,.$$

Conversely, suppose that $\liminf \frac{(\lambda_n)^j}{n^j} = 0$. As in [[9], p. 510], we can choose a subsequence $(n\,(p))_{p=1}^{\infty}$ such that $\frac{(\lambda_n(p))^j}{n(p)^j} < \frac{1}{p}$. Let $X = \mathbb{R}$ and g be the following metric: $g : \mathbb{R}^3 \to \mathbb{R}^+$, $g\,(x, y, z) = \max\left\{|x-y|, |x-z|, |y-z|\right\}$. We define a sequence $x = (x_n)$ by

$$x_n = \begin{cases} 1 & \text{if } i_1, i_2, ..., i_j \in I_{n(p)}, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $x \in g\,[C, 1]$, and therefore, by [[6], Theorem 2.1], $x \in gS$. However, on the other hand, $x \notin g\,[V, \lambda]$ and Theorem 3.7(ii) implies that $x \notin gS_\lambda$. Hence, $\liminf \frac{(\lambda_n)^j}{n^j} > 0$ is a necessary condition. □

Let σ be a mapping from the positive integers to themselves. A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or a σ-mean if it satisfies the following conditions:

(i) $\varphi\,(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(ii) $\varphi\,(e) = 1$, where $e = (1, 1, 1, ...)$ and

(iii) $\varphi\left(x_{\sigma(n)}\right) = \varphi\,(x_n)$ for all $x \in \ell_\infty$, where ℓ_∞ denotes the set of bounded sequences.

The mapping σ is assumed to be one-to-one and satisfies $\sigma^m(n) \neq n$ for all $n, m \in \mathbb{Z}^+$, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus, the functional φ extends the limit functional on c, the space of convergent sequences, in the sense that $\varphi(x_n) = \lim x_n$ for all $x \in c$. In the case where σ is the translation mapping $\sigma(n) = n+1$, the σ-mean is often referred to as a Banach limit. The space V_σ, which consists of bounded sequences whose invariant means are equal, can be shown to satisfy the following property:

$$V_\sigma = \left\{x \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} x_{\sigma^k(n)} = L, \text{ uniformly in } m\right\}.$$

In [19], Schaefer proved that a bounded sequence $x = (x_k)$ of real numbers is σ-convergent to L if and only if the following condition holds

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} x_{\sigma^i(m)} = L,$$

uniformly in m.
A sequence $x = (x_k)$ is said to be strongly σ-convergent to L if there exists a number L such that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} d\left(x_{\sigma^i(m)}, L\right) = 0,$$

as $k \to \infty$, uniformly in m. We denote the set of all strongly σ-convergent sequences as $[V_\sigma]$.
A sequence $x = (x_n) \in \ell_\infty$ is said to be almost convergent if all of its Banach limits coincide. The spaces of almost convergent sequences and strongly almost convergent sequences are defined respectively by

$$\widehat{c} = \left\{x \in \ell_\infty : \lim_m t_{mn}(x) \text{ exists uniformly in } n\right\}$$

and

$$[\widehat{c}] = \left\{x \in \ell_\infty : \lim_m t_{mn}(|x - le|) \text{ exists uniformly in } n \text{ for some } l \in \mathbb{C}\right\}$$

where $t_{mn}(x) = \frac{x_n + x_{n+1} + \ldots + x_{n+m}}{m+1}$ and $e = (1, 1, \ldots)$. Taking $\sigma(m) = m + 1$, we obtain $[V_\sigma] = [\widehat{c}]$.

Definition 3.9. *A sequence $x = (x_n)$ in a g-metric space (X, g) is said to be S_σ-convergent to x if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{j!}{n^j} \left|\left\{i_1, i_2, \ldots, i_j \leq n : g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, \ldots, x_{\sigma^{i_j}(m)}, x\right) \geq \varepsilon\right\}\right| = 0,$$

uniformly in m. In this case, we write $g\widehat{S}_\sigma$-$\lim x_n = x$ or $x_n \to x \left(g\widehat{S}_\sigma\right)$.

Before presenting the promised inclusion relations, we will provide a new definition.

Definition 3.10. *A sequence $x = (x_n)$ in a g-metric space (X, g) is said to be $S_{\sigma,\lambda}$-convergent to x if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{j!}{(\lambda_n)^j} \left|\left\{i_1, i_2, \ldots, i_j \in I_n : g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, \ldots, x_{\sigma^{i_j}(m)}, x\right) \geq \varepsilon\right\}\right| = 0,$$

uniformly in m. In this case, we write $g\widehat{S}_{\sigma,\lambda}$-$\lim x_n = x$ or $x_n \to x \left(g\widehat{S}_{\sigma,\lambda}\right)$.

Definition 3.11. *A sequence $x = (x_n)$ in a g-metric space (X, g) is said to be strongly $g(V, \lambda)$-summable to a number x if*

$$\lim_n \frac{j!}{(\lambda_n)^j} \sum_{i_1, i_2, \ldots, i_j \in I_n} g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, \ldots, x_{\sigma^{i_j}(m)}, x\right) = 0$$

uniformly in $m = 1, 2, 3, \ldots$, (denoted by $g\widehat{V}_{\sigma,\lambda}$-$\lim x_n = x$ or $x_n \to x \left(g\widehat{V}_{\sigma,\lambda}\right)$).

Now we give some inclusion relations between $g\widehat{S}_{\sigma,\lambda}$ and $g\widehat{V}_{\sigma,\lambda}$.

Theorem 3.12. *The following statements hold:*

(i) If $x_n \to x\left(g\widehat{V}_{\sigma,\lambda}\right)$ then $x_n \to x\left(g\widehat{S}_{\sigma,\lambda}\right)$,

(ii) If $x \in \ell_\infty$ and $x_n \to x\left(g\widehat{S}_{\sigma,\lambda}\right)$ then $x_n \to x\left(g\widehat{V}_{\sigma,\lambda}\right)$, and hence $x_n \to x\left(g\left[C,1\right]\right)$.

Proof. (i) Let $\varepsilon > 0$ and $x_n \to x\left(g\widehat{V}_{\sigma,\lambda}\right)$. Then we can write

$$\begin{aligned}
&\sum_{i_1,i_2,...,i_j \in I_n} g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \\
&\geq \sum_{\substack{i_1,i_2,...,i_j \in I_n \\ g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \geq \varepsilon}} g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \\
&\geq \varepsilon \left|\left\{i_1, i_2, ..., i_j \in I_n : g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \geq \varepsilon\right\}\right|.
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{j!}{\varepsilon . \left(\lambda_n\right)^j} \sum_{i_1,i_2,...,i_j \in I_n} g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \\
&\geq \frac{j!}{\left(\lambda_n\right)^j} \left|\left\{i_1, i_2, ..., i_j \in I_n : g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \geq \varepsilon\right\}\right|.
\end{aligned}$$

Hence, we obtain $x_n \to x\left(g\widehat{S}_{\sigma,\lambda}\right)$.

(ii) Suppose that $x \in \ell_\infty$ and $x_n \to x\left(g\widehat{S}_{\sigma,\lambda}\right)$. If $x \in \ell_\infty$, then, we can assume that

$$g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \leq M \text{ for all } j \text{ and } m.$$

Given $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{j!}{\left(\lambda_n\right)^j} \sum_{i_1,i_2,...,i_j \in I_n} g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \\
&= \frac{j!}{\left(\lambda_n\right)^j} \sum_{\substack{i_1,i_2,...,i_j \in I_n \\ g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \geq \varepsilon}} g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \\
&+ \frac{j!}{\left(\lambda_n\right)^j} \sum_{\substack{i_1,i_2,...,i_j \in I_n \\ g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) < \varepsilon}} g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \\
&\leq M \frac{j!}{\left(\lambda_n\right)^j} \left|\left\{i_1, i_2, ..., i_j \in I_n : g\left(x_{\sigma^{i_1}(m)}, x_{\sigma^{i_2}(m)}, ..., x_{\sigma^{i_j}(m)}, x\right) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}.
\end{aligned}$$

As a result, we conclude that $x_n \to x\left(g\widehat{V}_{\sigma,\lambda}\right)$. Additionally, we obtain

$$\begin{aligned}
&\tfrac{j!}{n^j}\sum_{i_1,i_2,...,i_j=1}^{n} g\left(x_{\sigma^{i_1}(m)},x_{\sigma^{i_2}(m)},...,x_{\sigma^{i_j}(m)},x\right)\\
&=\tfrac{j!}{n^j}\sum_{i_1,i_2,...,i_j=1}^{n-\lambda_n} g\left(x_{\sigma^{i_1}(m)},x_{\sigma^{i_2}(m)},...,x_{\sigma^{i_j}(m)},x\right)\\
&+\tfrac{j!}{n^j}\sum_{i_1,i_2,...,i_j\in I_n} g\left(x_{\sigma^{i_1}(m)},x_{\sigma^{i_2}(m)},...,x_{\sigma^{i_j}(m)},x\right)\\
&\le \tfrac{j!}{(\lambda_n)^j}\sum_{i_1,i_2,...,i_j=1}^{n-\lambda_n} g\left(x_{\sigma^{i_1}(m)},x_{\sigma^{i_2}(m)},...,x_{\sigma^{i_j}(m)},x\right)\\
&+\tfrac{j!}{(\lambda_n)^j}\sum_{i_1,i_2,...,i_j\in I_n} g\left(x_{\sigma^{i_1}(m)},x_{\sigma^{i_2}(m)},...,x_{\sigma^{i_j}(m)},x\right)\\
&\le \tfrac{2j!}{(\lambda_n)^j}\sum_{i_1,i_2,...,i_j\in I_n} g\left(x_{\sigma^{i_1}(m)},x_{\sigma^{i_2}(m)},...,x_{\sigma^{i_j}(m)},x\right).
\end{aligned}$$

Hence $x_n \to x\,(g\,[C,1])$, since $x_n \to x\left(g\widehat{V}_{\sigma,\lambda}\right)$. □

Theorem 3.13. *If*

$$\liminf \frac{(\lambda_n)^j}{n^j} > 0 \tag{3.1}$$

then

$$g\widehat{S}_\lambda - \lim x_n = x \textit{ implies } g\widehat{S}_{\sigma,\lambda} - \lim x_n = x.$$

Proof. For any given $\varepsilon > 0$ we get

$$\begin{aligned}
&\left\{i_1,i_2,...,i_j\in I_n : g\left(x_{\sigma^{i_1}(m)},x_{\sigma^{i_2}(m)},...,x_{\sigma^{i_j}(m)},x\right)\ge\varepsilon\right\}\\
&\subset\left\{i_1,i_2,...,i_j\le n : g\left(x_{\sigma^{i_1}(m)},x_{\sigma^{i_2}(m)},...,x_{\sigma^{i_j}(m)},x\right)\ge\varepsilon\right\}.
\end{aligned}$$

This gives

$$\begin{aligned}
&\frac{j!}{n^j}\left|\left\{i_1,i_2,...,i_j\le n : g\left(x_{\sigma^{i_1}(m)},x_{\sigma^{i_2}(m)},...,x_{\sigma^{i_j}(m)},x\right)\ge\varepsilon\right\}\right|\\
&\ge\frac{j!}{n^j}\left|\left\{i_1,i_2,...,i_j\in I_n : g\left(x_{\sigma^{i_1}(m)},x_{\sigma^{i_2}(m)},...,x_{\sigma^{i_j}(m)},x\right)\ge\varepsilon\right\}\right|\\
&\ge\frac{(\lambda_n)^j}{n^j}.\frac{j!}{(\lambda_n)^j}\left|\left\{i_1,i_2,...,i_j\in I_n : g\left(x_{\sigma^{i_1}(m)},x_{\sigma^{i_2}(m)},...,x_{\sigma^{i_j}(m)},x\right)\ge\varepsilon\right\}\right|.
\end{aligned}$$

Taking the limit as $n\to\infty$ and using equation (3.1), we obtain the desired result. This concludes the proof. □

If we set $\sigma(n) = n+1$ in the aforementioned Definitions 3.9, 3.10, and 3.11, we obtain the following definitions:

Definition 3.14. *A sequence $x = (x_n)$ in a g-metric space (X, g) is said to be almost statistically convergent to x provided that for all $\varepsilon > 0$,*

$$\lim_{n\to\infty}\frac{j!}{n^j}\left|\left\{i_1,i_2,...,i_j\le n : g\left(x_{i_1+m},x_{i_2+m},...,x_{i_j+m},x\right)\ge\varepsilon\right\}\right| = 0,$$

uniformly in m.

In this case, we write $g\widehat{S}\text{-}\lim x_n = x$ or $x_n \to x\left(g\widehat{S}\right)$.

Definition 3.15. *A sequence $x = (x_n)$ in a g-metric space (X, g) is said to be almost λ-statistically convergent to x provided that for all $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{j!}{(\lambda_n)^j} \left|\left\{i_1, i_2, ..., i_j \in I_n : g\left(x_{i_1+m}, x_{i_2+m}, ..., x_{i_j+m}, x\right) \geq \varepsilon\right\}\right| = 0,$$

uniformly in m.

In this case, we write $g\widehat{S}_\lambda\text{-}\lim x_n = x$ or $x_n \to x\left(g\widehat{S}_\lambda\right)$.
If $\lambda_n = n$, for all n, then $g\widehat{S}_\lambda$ is same as $g\widehat{S}$.

Definition 3.16. *A sequence $x = (x_n)$ in a g-metric space (X, g) is said to be strongly almost λ-summable to a number x if*

$$\lim_n \frac{j!}{(\lambda_n)^j} \sum_{i_1, i_2, ..., i_j \in I_n} g\left(x_{i_1+m}, x_{i_2+m}, ..., x_{i_j+m}, x\right) = 0$$

uniformly in $m = 1, 2, 3, ...,$ (denoted by $g\left[\widehat{V}_\lambda\right]\text{-}\lim x_n = x$ or $x_n \to x\left(\left[g\widehat{V}_\lambda\right]\right)$).

Remark. *Similar inclusions to Theorems 3.12 and 3.13 hold between strongly λ-almost statistically convergent and almost λ-statistically convergent.*

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