Ilirias Journal of Mathematics
ISSN: 2334-6574, URL: http://www.ilirias.com
Volume 1 Issue 1(2012), Pages 23-32.

APPLICATION OF (E. A.) PROPERTY VIA RATIONAL INEQUALITY IN MENGER SPACES

DEEPAK SINGH, AMIN AHMED, P.C. DIXIT

Abstract. In this paper, we obtain common fixed point theorems for rational inequality using common property (E. A.) . Our results generalize several known results in Menger spaces. Some related results are also derived besides furnishing an illustrative example.

1. Introduction

Menger [10] introduced the notion of probabilistic metric spaces as a generalization of core notion of metric space. In fact, he replaced the distance function $d : X \times X \to R^+$ with a distribution function $F_{x,y} : R \to [0, 1]$ where in for any number t, the value $F_{x,y}(t)$ describes the probability that the distance between x and y is less than t. The study of such spaces received an impetus with the pioneering works of Schweizer and Sklar [3]. The theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis due to its extensive applications in random differential as well as random integral equations. By now, several authors have already studied fixed point and common fixed point theorems in PM spaces which include [[4],[7],[9], [14], [15],[16],[19],[20],[21],[22]]. The study of common fixed points of non-compatible mappings is also equally interesting which has been initiated by Pant [18]. Recently, Aamri and Moutawakil [1] and Liu et al. [24] respectively defined the property (E.A) and common property (E.A) and proved some common fixed point theorems in metric spaces. Imdad et al. [13] extended the results of Aamri and Moutawakil [1] to semi-metric spaces. Most recently, Kubiaczyk and Sharma [7] defined the property (E.A) in PM spaces and used the same to prove some results on common fixed points wherein authors claim their results for strict contractions which are in fact proved for contractions. Ali et al.[8] introduced the notion of common property (E.A) in Menger spaces and utilized the same to prove some common fixed point theorems in Menger spaces. Inspired by the result Ali et al.[8]. We obtain common fixed point theorems for rational inequality using property (E.A.) and common property (E.A.). Our result generalize many known results in Menger spaces. Some related results are also derived besides furnishing an illustrative example.

2000 *Mathematics Subject Classification.* 54H25, 47H10.
Key words and phrases. Menger space,Property (E.A),Common property (E.A), Weakly compatible mappings and t-norm.

2. Priliminary estimates

Definition 2.1. *A mapping $F : R \to R^+$ is called distribution function if it is non-decreasing, left continuous with $inf\{F(t) : t \in R^+ = 0$ and $sup\{F(t) : t \in R^+ = 1$. Let L be the set of all distribution functions whereas H stands for the specific distribution function (also known as Heaviside function) defined by*

$$H(x) = \begin{cases} 0\ ; & x \leq 0 \\ 1\ ; & x > 0 \end{cases}$$

Definition 2.2. [10] *Let X be a non-empty set. An ordered pair (X, F) is called a PM space where F is a mapping from $X \times X$ into L satisfying the following conditions:*
(i) $F_{x,y}(t) = H(x)$ *if and only if* $x = y$;
(ii) $F_{x,y}(t) = F_{y,x}(t)$;
(iii) $F_{x,y}(t) = 1$ *and* $F_{y,z}(s) = 1$, *then* $F_{x,y}(t + s) = 1$, *for all* $x, y, x \in X$ *and* $t, s \geq 0$.

Every metric space (X, d) can always be realized as a PM space by considering $F : X \times X \to L$. L defined by $F_{x,y} = H(t - d(x, y))$ for all $x, y \in X$. So PM spaces offer a wider framework (than that of the metric spaces) and are general enough to cover even wider statistical situations.

Definition 2.3. [15] *A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if*
(i) $\Delta(a, 1) = a$, $\Delta(0, 0) = 0$;
(ii) $\Delta(a, b) = \Delta(b, a)$;
(iii) $\Delta(c, d) \geq \Delta(b, a)$ *for* $c \geq a, d \geq b$;
(iv) $\Delta(\Delta(a, b)c) = \Delta(a, \Delta(b, c))$ *for all* $a, b, c \in [0, 1]$.

Example 2.1. *The following are the four basic t-norms:*
(i) The minimum t-norm: $T_M(a, b) = min\{a, b\}$.
(ii) The product t-norm: $T_P(a, b) = a.b$.
(iii) The Lukasiewicz t-norm: $T_L(a, b) = max\{a + b - 1\}$.
(iv) The weakest t-norm, the drastic product:

$$H\,(x) = \begin{cases} min(a, b)\ if\ max(a, b) = 1 \\ 0\ otherwise \end{cases}$$

In respect of above mentioned t-norms, we have the following ordering :
$T_D < T_L < T_P < T_M$ Throughout this paper, Δ stands for an arbitrary continuous t-norm.

Definition 2.4. [3] *A Menger PM space is a triplet (X, F, Δ) where (X, F) is a PM space and Δ is a t-norm satisfying the following condition*
$F_{x,y}(t + s) \geq \Delta(F_{x,y}(t), F_{x,y}(s))$.

Definition 2.5. *A sequence $\{x_n\}$ in a Menger PM space (X, F, Δ) is said to converge to a point x in X if for every $\epsilon > 0$ and $\lambda > 0$, there is an integer $M(\epsilon, \lambda)$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$, for all $n \geq M(\epsilon, \lambda)$.*

Definition 2.6. [20] *A pair (A, S) of self mappings of a Menger PM space (X, F, Δ) is said to be compatible if $F_{ASx_n,SAx_n}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $AX_x, Sx_n \to t$, for some t in X as $n \to \infty$.*

Definition 2.7. *A pair (A, S) of self mappings of a Menger PM space (X, F, Δ) is said to be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that $AX_x, Sx_n \to t$ for some t in X as $n \to \infty$, but $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t_0)$ is either less than 1 or non existent, for some $t_0 > 0$.*

Definition 2.8. [7] *A pair (A, S) of self mappings of a Menger PM space (X, F, Δ) is said to satisfy the property (E.A) if there exist a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \quad \text{for some} \quad t \in X.$$

Clearly, a pair of non-compatible (as well as nontrivial compatible) mappings satisfies the property (E.A).

Definition 2.9. [8] *Two pairs (A, S) and (B, T) of self mappings of a Menger PM space (X, F, Δ) are said to satisfy the common property (E.A) if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X and some t in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = t \qquad .$$

Example 2.2. [8] *Let (X, F, Δ) be a Menger PM space with $X = [-1, 1]$ and*

$$F_{x,y}(t) = \begin{cases} e^{\frac{-|x-y|}{2}} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

for all $x, y \in X$. Define self mappings A, S, B and T on X as $Ax = \frac{x}{2}$, $Bx = \frac{-x}{2}$, $Sx = \frac{x}{4}$ and $Tx = \frac{-x}{4}$ for all $x \in X$. Then with sequences $\{x_n\} = \frac{1}{n}$ and $\{y_n\} = \frac{-1}{n}$ in X, one can easily verify that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = 0$. This shows that the pairs (A, S) and (B, T) share the common property (E.A.).

Definition 2.10. [15] *A pair (A, S) of self mappings of a nonempty set X is said to be weakly compatible if the mappings commute at their coincidence points, i.e. $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$.*

Lemma 2.1. [4] *Let (X, F, Δ) be a Menger space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, $F_{x,y}(kt) = F_{x,y}(t)$ then $x = y$.*

3. Main Results

Lemma 3.1. *Let A, B, S and T be self mappings of Menger spaces (x, F, Δ) satisfying the following conditions:*

(i) The pair (A, S) (or (B, T)) satisfies the properly (E.A.);

(ii) For any $x, y \in X$ and for all $t > 0$

$$\begin{aligned} F_{Ax,By}(kt) \geq min\Big\{ & F_{Sx,Ty}(t), F_{By,Ty}(t), F_{Ax,Ty}(t), \frac{F_{Ax,Ty}(t) * F_{By,Sx}(t)}{F_{Ax,Sx}(t)}, \\ & \frac{2.F_{Sx,Ty}(t)}{F_{Sx,Ty}(t) + F_{Ax,Ty}(t)}, 2.F_{Sx,Ty}(t)\Big\{\frac{1 + F_{Ax,Sx}(t)}{1 + F_{By,Ty}(t)}\Big\}\Big\} \end{aligned} \tag{3.1}$$

Where $0 < k < 1$;

(iii) $A(X) \subset T(X)$, $B(X) \subset S(X)$. Then the pair (A, S) and (B, T) share the common property (E.A.).

Proof: Suppose that the pair (A, S) possesses property (E.A.), then there exists a sequence $\{x_n\}$ in X such that, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = l$.

Since $A(X) \subset T(X)$, hence for each $\{x_n\}$ there exists $\{y_n\} \in X$ such that $Ax_n = Ty_n$. Therefore $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Sx_n = l$. Thus in all, we have $Ax_n \to l$, $Sx_n \to l$

and $Ty_n \to l$.
Now we asserts that $By_n \to l$ using inequality (3.1) with $x = x_n$, $y = y_n$. We have

$$F_{Ax_n,By_n}(kt) \geq min\Big\{F_{Sx_n,Ty_n}(t), F_{By_n,Ty_n}(t), F_{Ax_n,Ty_n}(t), \frac{F_{Ax_n,Ty_n}(t) * F_{By_n,Ty_n}(t)}{F_{Ax_n,Sx_n}(t)},$$
$$\frac{2.F_{Sx_n,Ty_n}(t)}{F_{Sx_n,Ty_n}(t) + F_{Ax_n,Ty_n}(t)}, 2.F_{Ax_n,Ty_n}(t)\Big\{\frac{1 + F_{Ax_n,Sx_n}(t)}{1 + F_{By_n,Ty_n}(t)}\Big\}\Big\}$$

Which on making $n \to \infty$ reduces to

$$F_{l,By_n}(kt) \geq min\{1, F_{l,By_n}(t), 1, \frac{1 * F_{l,By_n}(t)}{1}, 1, 2.1\Big\{\frac{1+1}{1 + F_{l,By_n}(t)}\Big\}\}$$
$$\geq F_{l,By_n}(t).$$

Which amounts to say That $By_n \to l$ using lemma (2.1). Thus, we have shown that the pair (A, S) and (B, T) share the common property (E.A.).

Remark. *The converse of lemma (3.1) is not true in general for counter example, one can utilize example (3.1).*

Result for common property (E.A.)

Theorem 3.2. *Let A, B, S and T be self mappings of a Menger space (X, F, Δ) with satisfying the inequality (3.1). Suppose that*
(i) The pairs (A, S) and (B, T) share the common property (E.A.);
(ii) $S(X)$ and $T(X)$ are closed subsets of X. Then the pairs (A, S) and (B, T) have a coincidence point each. Moreover A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof:- In view of (i) there exists two sequences x_n and y_n in X such that
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = l, \text{ for some } l \in X$$
Since $S(X)$ is a closed subset of X, therefore $\lim_{n\to\infty} Sx_n = l \in S(X)$. Which amounts to say that there exists a point $u \in X$ such that $Su = l \in S(X)$. Now we assert that $Au = Su$, to prove this, using inequality (3.1) with $x = u$, $y = y_n$ One gets

$$F_{Au,By_n}(kt) \geq min\{F_{Su,Ty_n}(t), F_{Su,By_n}(t), F_{Au,Ty_n}(t), \frac{F_{Au,Ty_n}(t) * F_{By_n,Ty_n}(t)}{F_{Au,Su}(t)},$$
$$\frac{2.F_{Su,Ty_n}(t)}{F_{Su,Ty_n}(t) + F_{Au,Ty_n}(t)}, 2.F_{Au,Ty_n}(t)\Big\{\frac{1 + F_{Au,Su}(t)}{1 + F_{Su,By_n}(t)}\Big\}\Big\}$$

Which on making $n \to \infty$ reduces to

$$F_{Au,l}(kt) \geq min\{1, 1, F_{Au,l}(t), \frac{F_{Au,l}(t) * 1}{F_{Au,l}(t)}, \frac{2.1(t)}{1 + F_{Au,l}(t)}, 2.F_{Au,l}(t)\Big\{\frac{1 + F_{Au,l}(t)}{1+1}\Big\}\Big\}$$
$$\geq F_{Au,l}(t).$$

Owing to Lemma (2.1)we have $Au = l$ and hence $Au = Su$ which shows that u is a coincidence point of the pair (A, S). Since $T(X)$ is also a closed subset of X, therefore $\lim_{n\to\infty} Ty_n = l \in T(X)$ and hence one can find a point $w \in X$ such that $Tw = l$. Now we show that $Bw = Tw$. To accomplish this, on using inequality

(3.1) with $x = u$, $y = w$, we have

$$F_{Au,Bw}(kt) \geq min\{F_{Su,Tw}(t), F_{Su,Bw}(t), F_{Au,Tw}(t), \frac{F_{Au,Tw}(t) * F_{Bw,Tw}(t)}{F_{Au,Su}(t)}, \frac{2.F_{Su,Tw}(t)}{F_{Su,Tw}(t) + F_{Au,Tw}(t)}, 2.F_{Au,Tw}(t)\Big\{\frac{1 + F_{Au,Su}(t)}{1 + F_{Su,Bw}(t)}\Big\}\}$$

$$F_{l,Bw}(kt) \geq Min\{1, F_{l,Bw}(t), 1, \frac{1 * F_{l,Bw}(t)}{1}, \frac{2.1}{1+1}, 2.1\Big\{\frac{1+1}{1 + F_{l,Bw}}\Big\}\} \geq F_{l,Bw}(t).$$

On employing Lemma (2.1), we have $Bw = l$ and hence $Tw = Bw$ which shows that w is a coincidence point of the pair (B, T).
Since the pair (A, S) is weakly compatible and $Au = Su$, Therefore $Al = ASu = SAu = Sl$. Now on using inequality (3.1) with $x = l$, $y = w$, We have

$$F_{Al,Bw}(kt) \geq min\{F_{Sl,Tw}(t), F_{Sl,Bw}(t), F_{Al,Tw}(t), \frac{F_{Al,Tw}(t) * F_{Bw,Tw}(t)}{F_{Al,Sl}(t)}, \frac{2.F_{Sl,Tw}(t)}{F_{Sl,Tw}(t) + F_{Al,Tw}(t)}, 2.F_{Al,Tw}(t)\Big\{\frac{1 + F_{Al,Sl}(t)}{1 + F_{Sl,Bw}(t)}\Big\}\}$$

$$F_{Al,l}(kt) \geq min\{F_{Al,l}(t), F_{Al,l}(t), F_{Al,l}(t), \frac{F_{Al,l}(t) * 1}{1}, \frac{2.F_{Al,l}(t)}{F_{Al,l}(t) + F_{Al,l}(t)}, 2.F_{Al,l}(t)\Big\{\frac{1+1}{1 + F_{Al,l}(t)}\Big\}\}$$

Or $F_{Al,l}(kt) \geq F_{Al,l}(t)$.
Appealing to Lemma (2.1), we have $Al = Sl = l$ which shows that l is a common fixed point of the pair (A, S). Also the pair (B, T) is weakly compatible and $Tw = Bw$, hence $Bl = BTw = TBw = Tl$.
Next we assert that l is also a common fixed point of the pair (B, T). ln order to establish this, using inequality (3.1) with $x = u$, $y = l$ We have

$$F_{Au,Bl}(kt) \geq min\{F_{Su,Tl}(t), F_{Su,Bl}(t), F_{Au,Tl}(t), \frac{F_{Au,Tl}(t) * F_{Bl,Tl}(t)}{F_{Au,Su}(t)}, \frac{2.F_{Su,Tl}(t)}{F_{Su,Tl}(t) + F_{Au,Tl}(t)}, 2.F_{Au,Tl}(t)\Big\{\frac{1 + F_{Au,Su}(t)}{1 + F_{Su,Bl}(t)}\Big\}\}$$

$$F_{l,Bl}(kt) \geq min\{F_{l,Bl}(t), F_{Bl,l}(t), F_{Bl,l}(t), \frac{F_{Bl,l}(t) * 1}{1}, \frac{2.F_{l,Bl}(t)}{F_{l,Bl}(t) + F_{Bl,l}(t)}, 2.F_{Bl,l}(t)\Big\{\frac{1+1}{1 + F_{l,Bl}(t)}\Big\}\}$$

Or $F_{l,Bl}(kt) \geq F_{l,Bl}(t)$.
In view of lemma (2.1), we have $Bl = l$ which shows that l is a common fixed point of the pair (B, T) and in all l is a common fixed point of both the pairs (A, S) and (B, T). The uniqueness of the common fixed point follows from inequality (3.1).

Remark. *Theorem (3.2) also extend the main result of Kubiaczyk and Sharma* [7] *to two pairs of mappings without any condition on containment of ranges amongst involved mappings.*

Theorem 3.3. *The conclusion of Theorem (3.2) remain true if the condition (ii) of Theorem (3.2) is replaced by the following:*
(iii) $\overline{A(X)} \subset T(X)$ *and* $\overline{B(X)} \subset S(X)$.

Corollary 3.4. *The conclusions of Theorem (3.2) and Theorem (3.3) remain true if the condition (ii) and (iii) are replaced by the following:*
(iv) $A(X)$ *and* $B(X)$ *are closed subset of* X *provided* $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$.

Theorem 3.5. *Let* A, B, S *and* t *be self mappings of a Menger spaces* (X, F, Δ) *with satisfying the inequality (3.1) and the conditions:*
(i) The pair (A, S) *or* (B, T) *enjoys the property (E.A.)*
(ii) $A(X) \subset T(X)$ *(or* $B(X) \subset S(X)$*);*
(iii) $S(X)$ *(or* $T(X)$*) is closed subset of* X.

Then the pair (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided both the pair (A, S) and (B, T) are weakly compatible.
Proof:- In view of Lemma (3.1), the pairs (A, S) and (B, T) share the common properly (E.A.) i.e. there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = l \text{ for some } l \in X$$
.

If $S(X)$ is a closed subset of X, then on the lines of Theorem (3.2), one can show that the pair (A, S) has coincidence point, say u, i.e., $Au = Su = l$. Since $A(X) \subset T(X)$ and $Au \in A(X)$, there exists $w \in X$ such that $Au = Tw$. Now we assert that $Bw = Tw$. On using inequality (3.1) with $x = u$ and $y = w$, one gets

$$F_{Au,Bw}(kt) \geq Min\{F_{Su,Tw}(t), F_{Bw,Su}(t), F_{Au,Tw}(t), \frac{F_{Au,Tw}(t) * F_{Bw,Tw}(t)}{F_{Au,Su}(t)}, \frac{2.F_{Su,Tw}(t)}{F_{Su,Tw}(t) + F_{Au,Tw}(t)}, 2.F_{Au,Tw}(t)\Big\{\frac{1 + F_{Au,Su}(t)}{1 + F_{Bw,Su}(t)}\Big\}\}$$

$$F_{l,Bw}(kt) \geq Min\{F_{l,Tw}(t), F_{l,Bw}(t), F_{l,Tw}(t), \frac{F_{l,Tw}(t) * F_{Bw,Tw}(t)}{F_{l,l}(t)}, \frac{2.F_{l,Tw}(t)}{F_{l,Tw}(t) + F_{l,Tw}(t)}, 2.F_{l,Tw}(t)\Big\{\frac{1 + F_{l,l}(t)}{1 + F_{l,Bw}(t)}\Big\}\}$$

$$F_{Tw,Bw}(kt) \geq Min\{1, F_{Tw,Bw}(t), 1, \frac{1 * F_{Tw,Bw}(t)}{1}, \frac{2.1}{1+1}, 2.1\Big\{\frac{1+1}{1 + F_{Tw,Bw}(t)}\Big\}\}$$

Or $F_{Tw,Bw}(kt) \geq F_{Tw,Bw}(t)$
On using Lemma (2.1), we have $Tw = Bw$, which shows that w, is a coincidence point of the pair (B, T). Since the pair (A, S) is weakly compatible and $Au = Su$, Therefore $ASu = SAu$ that is $Al = Sl$. Now on using inequality (3.1) with $x = l$, $y = w$. We have

$$F_{Al,Bw}(kt) \geq Min\{F_{Sl,Tw}(t), F_{Bw,Sl}(t), F_{Al,Tw}(t), \frac{F_{Al,Tw}(t) * F_{Bw,Tw}(t)}{F_{Al,Sl}(t)}, \frac{2.F_{Sl,Tw}(t)}{F_{Sl,Tw}(t) + F_{Al,Tw}(t)}, 2.F_{Al,Tw}(t)\Big\{\frac{1 + F_{Al,Sl}(t)}{1 + F_{Bw,Sl}(t)}\Big\}\}$$

$$F_{Al,l}(kt) \geq Min\{F_{Al,l}(t), F_{Al,l}(t), F_{Al,l}(t), \frac{F_{Al,l}(t) * 1}{1}, \frac{2.F_{Al,l}(t)}{F_{Al,l}(t) + F_{Al,l}(t)}, 2.F_{Al,l}(t)\Big\{\frac{1+1}{1+F_{Al,l}(t)}\Big\}\}$$

Or $F_{Al,l}(kt) \geq F_{Al,l}(t)$

On using lemma (2.1), we have $Al = Sl = l$ which shows that l is a coincidence point of the pair (A, S).

Corollary 3.6. *Let A and S be self mappings of a Menger space (X, F, Δ). Suppose that*

(i) The pair (A, S) enjoys the property (E.A.);

(ii) For all $x, y \in X$ and for all $t > 0$.

$$F_{Ax,Ay}(kt) \geq Min\{F_{Sx,Sy}(t), F_{Ay,Sx}(t), F_{Ax,Sy}(t), \frac{F_{Ax,Sy}(t) * F_{Ay,Sy}(t)}{F_{Ax,Sx}(t)}, \frac{2.F_{Sx,Sy}(t)}{F_{Sx,Sy}(t) + F_{Ax,Sy}(t)}, 2.F_{Ax,Sy}(t)\Big\{\frac{1 + F_{Ax,Sx}(t)}{1 + F_{Ay,Sx}(t)}\Big\}\} \tag{3.2}$$

Where $0 < k < 1$ (iii) $S(X)$ is a closed subset of X.

Then A and S have a coincidence point. Moreover, if the pair (A, S) is weakly compatible, then A and S have a unique common fixed point.

Proof: Proof follows on the similar lines as done in Theorem (3.5)

Our next result involves a lower semi continuous function $\phi : [0, 1] \to [0, 1]$ such that $\phi(t) > t$ for all $t \in (0, 1)$, $\phi(0) = 0$ and $\phi(1) = 1$.

Theorem 3.7. *Let A, B, S and T be self-mappings of a Menger space (X, F, Δ) satisfying conditions: (i) The pairs (A, S) and (B, T) share the common property (E.A.); (ii) $S(X)$ and $T(X)$ are closed subsets of X . (iii) for all $x, y \in X$ and $t > 0$.*

$$F_{Ax,By}(t) \geq \phi(min\{F_{Sx,Ty}(t), F_{By,Ty}(t), F_{Ax,Ty}(t), \frac{F_{Ax,Ty}(t) * F_{By,Sx}(t)}{F_{Ax,Sx}(t)}, \frac{2.F_{Sx,Ty}(t)}{F_{Sx,Ty}(t) + F_{Ax,Ty}(t)}, 2.F_{Ax,Ty}(t)\Big\{\frac{1 + F_{Ax,Sx}(t)}{F_{By,Ty}(t)}\Big\}\}) \tag{3.3}$$

Then the pair (A, S) and (B, T) have a point of coincidence each. Moreover A, B, S and T have unique common fixed point provided both the pair (A, S) and (B, T) are weakly compatible.

Proof:- As both the pairs share the common property (E.A.) there exists sequence $\{x_n\}, \{y_n\} \in X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = l \text{ for some } l \in X$$

.

if $S(X)$ is a closed subset of X, then $\lim_{n\to\infty} Sx_n = l \in S(X)$, therefore, there exists point $u \in X$ such that $Su = l$

Now we assert that $Au = Su$. Setting $x = u$, $y = \{y_n\}$ in inequality(3.3). We get

$$F_{Au,By_n}(t) \geq \phi(min\{F_{Su,Ty_n}(t), F_{By_n,Su}(t), F_{Au,Ty_n}(t), \frac{F_{Au,Ty_n}(t) * F_{By_n,Ty_n}(t)}{F_{Au,Su}(t)}, \frac{2.F_{Su,Ty_n}(t)}{F_{Su,Ty_n}(t) + F_{Au,Ty_n}(t)}, 2.F_{Au,Ty_n}(t)\Big\{\frac{1 + F_{Au,Su}(t)}{1 + F_{By_n,Su}(t)}\Big\}\})$$

Which on making $n \to \infty$, reduces to

$$\begin{aligned} F_{Au,l}(t) &\geq \phi(min\{1, 1, F_{l,Au}(t), \frac{F_{l,Au}(t) * 1}{1}, \frac{2.1}{1 + F_{l,Au}(t)}, 2.F_{l,Au}(t)\Big\{\frac{1 + F_{l,Au}(t)}{1+1}\Big\}\}) \\ &\geq \phi(F_{Au,l}(t)) \\ &> F_{Au,l}(t) \end{aligned}$$

Which is contradiction.
Therefore $Au = l$, and henceforth $Au = Su$ which shows that the pair (A, S) has a coincidence point. Again $T(X)$ is a closed subset of X, then $\lim_{n\to\infty} Ty_n = l \in T(X)$. Therefore, there exists a point $w \in X$ such that $Tw = l$. Now, we claim that $Bw = Tw$. To establish this, using inequality (3.3) with $x = u$, $y = w$, We have

$$\begin{aligned} F_{Au,Bw}(t) \geq \phi(min\{F_{Su,Tw}(t), F_{Bw,Su}(t), F_{Au,Tw}(t), \frac{F_{Au,Tw}(t) * F_{Bw,Tw}(t)}{F_{Au,Su}(t)}, \\ \frac{2.F_{Su,Tw}(t)}{F_{Su,Tw}(t) + F_{Au,Tw}(t)}, 2.F_{Au,Tw}(t)\Big\{\frac{1 + F_{Au,Su}(t)}{1 + F_{Bw,Su}(t)}\Big\}\}) \end{aligned}$$

Which on making $n \to \infty$, reduces to

$$\begin{aligned} F_{l,Bw}(t) &\geq \phi(min\{1, F_{l,Bw}(t), 1, \frac{1 * F_{l,Bw}(t)}{1}, \frac{2.1}{1+1}, 2.1\Big\{\frac{1+1}{1 + F_{l,Bw}(t)}\Big\}\}) \\ &\geq \phi(F_{l,Bw}(t)) \\ &> F_{l,Bw}(t) \end{aligned}$$

Which is contradiction.
Therefore $Bw = l$, and henceforth $Tw = Bw$ which shows that the pair (B, T) also has a coincidence point.
Since the pairs (A, S) and (B, T) are weakly compatible and both the pairs have point of coincidence u and w respectively. Hence we can easily prove the existence of unique common fixed point of mappings A, B, S and T.

Remark. *Theorem (3.7) partially generalizes the main result of Kohli and Vashistha* [9] *to two pair of self mappings as Theorem 3.4 never requires any condition on the containment of ranges among involved mapping besides weakening the completeness requirement of the space to closeness of the subsets. Here one may also notice that function ϕ is lower semi-continuous whereas all the involved mappings can be discontinuous at the time.*

Now we give an example to support our Theorem (3.2)

Example 3.1. *Consider $X = [-1, 1]$ and define $F_{x,y}(t) = H(t - |x - y|)$ for all $x, y \in X$. Then (X, F, Δ) is a Menger PM space with $\Delta(a, b) = Min\{a, b\}$. Define self-mappings A, B, S and T on X as*

$$A(x) = \begin{cases} \frac{3}{4} \; if \; x \in \{-1, 1\} \\ \frac{x}{5} \; if \; x \in (-1, 1) \end{cases}$$

$$B(x) = \begin{cases} \frac{3}{4} \; if \; x \in \{-1, 1\} \\ \frac{-x}{5} \; if \; x \in (-1, 1) \end{cases}$$

$$S(x) = \begin{cases} \frac{1}{2} \; if \; x = -1 \\ \frac{x}{2} \; if \; x \in (-1,1) \\ \frac{-1}{2} \; if \; x = 1 \end{cases}$$

$$T(x) = \begin{cases} \frac{-1}{2} \; if \; x = -1 \\ \frac{-x}{2} \; if \; x \in (-1,1) \\ \frac{1}{2} \; if \; x = 1 \end{cases}$$

Then with sequences $\{x_n = 1/n\}$ *and* $\{y_n = -1/n\}$ *in* X *we have* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n == 0$*, Which shows that pair* (A, S) *and* (B, T) *share the common property (E.A.). By a routine calculation, one can verify the contraction condition (3.1) with K=1/2. Also* $A(X) = B(X) = \{\frac{3}{4}\} \cup (\frac{-1}{5}, \frac{1}{5})$ *not proper subset of* $[\frac{-1}{2}, \frac{1}{2}] = S(X) = T(X)$

Thus, All the conditions of Theorem (3.2) are satisfied and 0 is unique common fixed point of pairs (A, S) and (B, T) which is their coincidence point as well.

Here it is worth noting that the majority of earlier established theorems (with rare possible exceptions) can not be used in the context of this example as theorem 2.1 never require any condition on the containment of ranges amongst the in valued mappings. Also the closeness of the subspaces replaces the completeness condition.

Remark. *The continuity requirements of the entire involved mapping are completely relaxed whereas most of earlier theorems require the continuity of at least one involved mappings.*

Acknowledgments. The author would like to thank to Dr. J. Ali for his paper [8] and grateful to an anonymous referee for his/her fruitful suggestions that helped us improve this paper.

References

[1] A. Aamri, D. El Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl. **270** (2002) 181-188.

[2] A. Razani, M. Shirdaryazdi, *A common fixed point theorem of compatible maps in Menger space*, Chaos Solitons Fractals **32** (2007) 26-34.

[3] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, Elsevier, North Holland, New York, 1983.

[4] B. Singh, S. Jain, *A fixed point theorem in Menger spaces through weak compatibility*, J. Math. Anal. Appl. **301** (2005) 439-448.

[5] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci. **4 (2)** (1996) 199-215.

[6] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9 (4)** (1986) 771-779.

[7] I. Kubiaczyk, S. Sharma, *Some common fixed point theorems in Menger space under strict contractive conditions*, Southeast Asian Bull. Math. **32** (2008) 117-124.

[8] J. Ali,M. Imdad,D. Bahuguna, *Common Fixed point theorems in Menger spaces with common property (E.A.)*,Computer And Mathematics With Applications **60**(2010) 3151-3159.

[9] J.K. Kohli, S. Vashistha, *Common fixed point theorems in probabilistic metric spaces*, Acta Math. Hungar. **11)** (2007) 37-47.

[10] K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci. USA **28** (1942) 535-537.

[11] Lj.B. Ciric, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974) 267-273.

[12] M. Imdad, J. Ali, *Jungck's common fixed point theorem and E.A property*, Acta Math. Sinica **24** (2008) 87-94.

[13] M. Imdad, J. Ali, L. Khan, *Coincidence and fixed points in symmetric spaces under strict contractions*, J. Math. Anal. Appl. **320** (2006) 352-360.
[14] M. Imdad, J. Ali, M. Tanveer, *Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces*, Chaos Solitons Fractals **42** (2009) 3121-3129.
[15] O. Hadzic, E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.
[16] R. Chugh, S. Rathi, *Weakly compatible maps in probabilistic metric spaces*, J. Indian Math. Soc. **72** (2005) 131-140.
[17] R.A. Rashwan, A. Hedar, *On common fixed point theorems of compatible maps in Menger spaces*, Demonst. Math. **31** (1998) 537-546.
[18] R.P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. **188** (1994) 436-440.
[19] S. Kutukcu, S. Sharma, *Compatible maps and common fixed points in Menger probabilistic metric spaces*, Commun. Korean Math. Soc. **24** (2009)17-27.
[20] S.N. Mishra, *Common fixed points of compatible mappings in PM-spaces*, Math. Japon. **36** (1991) 283-289.
[21] T.L. Hicks, *Fixed point theory in probabilistic metric spaces*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **13** (1983) 63-72.
[22] V.M. Sehgal, A.T. Bharucha-Reid, *Fixed point of contraction mappings on probabilistic metric spaces*, Math. Syst. Theory **6** (1972) 97-102.
[23] V.W. Bryant, *A remark on fixed point theorem for iterated mappings*, Amer. Math. Monthly **75** (1968) 399-400.
[24] Y. Liu, Jun Wu, Z. Li, *Common fixed points of single-valued and multi-valued maps*, Int. J. Math. Math. Sci. **19** (2005) 3045-3055.

Deepak Singh Department of Mathematics, NITTTR (Ministry of HRD,Govt. of India), Bhopal India. E-mail: deepak.singh.2006@indiatimes.com
E-mail address: deepak.singh.2006@indiatimes.com

Amin Ahmed
University Inst. Of Technology, Rajiv Gandhi Technical University, Bhopal, (M.P.), India, 462036.
E-mail address: aminkhan.iqra@gmail.com

P.C. Dixit
Technocrate Inst. of Technology, Near Anand Nagar, Bhopal, (M.P.), India, 462036.