Ilirias Journal of Mathematics
ISSN: 2334-6574, URL: http://www.ilirias.com
Volume 1 Issue 1(2012), Pages 14-22.

SOME RESULTS OF VERY WEAK SOLUTIONS FOR OBSTACLE PROBLEM OF NONHOMOGENEOUS

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Abstract. This paper gives the definition and obtain some properties of the very weak solutions of nonhomogeneous obstacle problem,in particular with the Quasiminimizers,the local and global higher integrability of the derivative. Here we used Poincaré inequality, Young inequality, Hölder inequality and Hodge decompose etc to prove these properties.

1. Introduction and Statement of Results

In this paper we consider the second order degenerate nonhomogeneous elliptic equation

$$divA(x,\nabla u(x)) = divF(x), \tag{1.1}$$

where Ω is a bounded regularity area in $R^n, n \geq 2$ in which the estimates of Hodge decompose:(4), (5)come into existence. $A(x,\xi) : \Omega \times R^n \to R^n$ is a Carathéodory function satisfying the structural conditions:

$$\begin{array}{ll} (i) & \langle A(x,\xi),\xi\rangle \geq \alpha|\xi|^p, \quad \forall\ \xi \in R^n, \\ (ii) & |A(x,\xi)| \leq \beta|\xi|^{p-1}, \quad \forall\ \xi \in R^n, \\ (iii) & \langle A(x,\xi_1) - A(x,\xi_2), \xi_1 - \xi_2\rangle > 0, \\ & \forall\ \xi_1, \xi_2 \in R^n, \xi_1 \neq \xi_2. \end{array}$$

where $p > 1, 0 < \alpha \leq \beta < \infty$ and $F(x) \in \left(L^{p'}_{loc}(\Omega)\right), p' = \frac{p}{p-1}$.

The prototype of equation (1) is the homogeneous A-harmonic equation

$$divA(x,\nabla u(x)) = 0. \tag{1.2}$$

Suppose that ψ is any function in Ω with values in $R \cup (-\infty,\infty)$, and that $\theta \in W^{1,r}(\Omega), max\{1,p-1\} < r \leq p$. Let

$$K^r_{\psi,\theta} = \{v \in W^{1,r}(\Omega) : v \geq \psi, \quad \text{a.e.}, \quad v - \theta \in W^{1,r}_0(\Omega)\}.$$

The function ψ is an obstacle and θ determines the boundary values.

We introduce the Hodge decomposition for $|\bigtriangledown(v-u)|^{r-p}\bigtriangledown(v-u) \in L^{\frac{r}{r-p+1}}(\Omega)$, see [1],

$$|\bigtriangledown(v-u)|^{r-p}\bigtriangledown(v-u) = \bigtriangledown\phi_{v,u} + h_{v,u}, \tag{1.3}$$

2000 *Mathematics Subject Classification.* 53D15, 53C21, 53C25, 53C40.

Key words and phrases. Invariant submanifolds; Sasakian manifold; Quarter symmetric metric connection; totally geodesic.

where $\phi_{v,u} \in W^{1,\frac{r}{r-p+1}}(\Omega)$ and $h_{v,u} \in L^{\frac{r}{r-p+1}}(\Omega)$ is divergence free vector.The following estimates hold

$$\| \bigtriangledown \phi_{v,u}\|_{\frac{r}{r-p+1}} \leq c\| \bigtriangledown (v-u)\|_r^{r-p+1}, \tag{1.4}$$

$$\|h_{v,u}\|_{\frac{r}{r-p+1}} \leq c(p-r)\| \bigtriangledown (v-u)\|_r^{r-p+1}. \tag{1.5}$$

The definition of very weak solution of obstacle problem for equation (2) is the following:

Definition 1 *A very weak solution to the $K^r_{\psi,\theta}$-obstacle problem is a function $u \in K^r_{\psi,\theta}$ such that*

$$\int_\Omega \langle A(x,\bigtriangledown u), |\bigtriangledown (v-u)|^{r-p} \bigtriangledown (v-u)\rangle dx \geq \int_\Omega \langle A(x,\bigtriangledown u), h_{v,u}\rangle dx, \tag{1.6}$$

whenever $v \in K^r_{\psi,\theta}$ and $h_{v,u}$ comes from the Hodge decomposition (3).

Now we give the definition of the very weak solution to obstacle problems for nonhomogeneous elliptic equation.

Definition 2 *A very weak solution to the $K^r_{\psi,\theta}$-obstacle problem for equation (1) is a function $u \in K^r_{\psi,\theta}$ such that*

$$\int_\Omega \langle A(x,\bigtriangledown u), |\bigtriangledown (v-u)|^{r-p} \bigtriangledown (v-u)\rangle dx$$
$$\geq \int_\Omega \langle A(x,\bigtriangledown u), h_{v,u}\rangle dx + \int_\Omega \langle F, \bigtriangledown \phi_{v,u}\rangle dx, \tag{1.7}$$

whenever $v \in K^r_{\psi,\theta}$ and $\bigtriangledown\phi_{v,u}, h_{v,u}$ come from the Hodge decomposition.

Many interesting results have been obtained for the solutions of (2).Meyers and Elcrat (see[2]) first considered the higher integrability of solutions of (2) in 1975. For the local and global higher integrability of the derivatives in obstacle problems , Li Gongbao and O.Martio first considered it in 1994 (see[3]). The higher integrability of very weak solutions for obstacle problems associated with equation (2) has been studied in [4]. Recently, we hade considered the higher integrability for the solutions and the local regularity for very weak solutions of obstacle problems of (1)(see[5],[6]). But the theory for the very weak solutions of obstacle problems of (1) have not been explored. This paper not only gives the definition but also gives some properties of the very weak solutions for $K_{\psi,\theta}$-obstacle problems associated with equation (1).

Here we consider the following properties for the very weak solutions of nonhomogeneous $K_{\psi,\theta}$-obstacle problem.

Theorem 1. *(Quasiminimizers) There exists $r_0 \in (p-1,p)$, such that for arbitrary $r > r_0$, the very weak solution u to the nonhomogeneous $K^r_{\psi,\theta}$-obstacle problem quasiminimizer the r-Dirichlet integral in $K^r_{\psi,\theta}(\Omega)$, that is,*

$$\int_\Omega |\bigtriangledown u|^r dx \leq c\left(\int_\Omega |\bigtriangledown v|^r dx + \int_\Omega |F|^{\frac{r}{p-1}} dx\right), \tag{1.8}$$

for every $v \in K^r_{\psi,\theta}$, where the constant C depending on n, r_0, p, α and β.

This theorem tells us the very weak solution of (1) quasiminimizer the r-Dirichlet integral in $K^r_{\psi,\theta}(\Omega)$. The same result of (2) see [4].

For the local and global higher integrability of the derivative, we have following two theorems.

Theorem 2. *(Local higher integrability) Suppose that* $F(x) \in \left(L_{loc}^{\frac{r'}{p-1}}(\Omega)\right)^n, r < r' < n$*, there exists* $r_1 \in (p-1, p)$*, such that while* $r_1 < r \le p$*, for arbitrary* $\psi \in W_{loc}^{1,s}(\Omega), s > r$*, a very weak solution* u *to the* $K_{\psi,\theta}^r$*-obstacle problem belongs to* $W_{loc}^{1,q}(\Omega)$*, where* $q > r$*.*

It seems necessary to impose a regularity condition for $\partial\Omega$ for studying the global higher integrability of the derivative ∇u(see[1]). We say that $\partial\Omega$ is $r-$ *Poincaré thick* if there is $\gamma < \infty$ such that for all open cubes $Q(R) \subset \mathbf{R}^n$ with side length $R > 0$ it holds

$$\left(\int_{Q(2R)} |u|^r dx\right)^{1/r} \le \gamma \left(\int_{Q(2R)} |\nabla u|^{\frac{rn}{r+n}} dx\right)^{(r+n)/rn}. \tag{1.9}$$

whenever $u \in W^{1,r}(Q(2R)), u = 0$ a.e. on $(\mathbf{R}^n \backslash \Omega) \cap Q(2R)$, and $Q(\frac{3}{2}R) \cap \mathbf{R}^n \backslash \Omega \neq \emptyset$.

Theorem 3. *(Global higher integrability) Suppose that a bounded domain* Ω *has a r-Poincaré thick boundary and that* $r \ge n/(n-1)$*. Let* θ *and* ψ *belong to* $W^{1,s}(\Omega), s > r$*. there exists* $r_2 \in (p-1, p)$*, such that while* $r_2 < r \le p$*, then a very weak solution* u *to the* $K_{\psi,\theta}^r$*-obstacle problem of equation (2) belongs to* $W^{1,q}(\Omega)$ *where* $q > r$ *.*

2. PRELIMINARY RESULTS

We now introduce some symbols and results used in this paper.

1. Sobolev-Poincaré inequality[7] Let Ω be a $C^{0,1}$ domain. For any function $u \in W^{1,p}(\Omega)$, $1 \le p < n$, the inequality

$$\left(\int_{\Omega} |u - c_u|^q dx\right)^{1/q} \le c(n,p) \left(\int_{\Omega} |\nabla u|^p dx\right)^{1/p} \tag{2.1}$$

holds with the universal constant $c(n,p)$, $q = pn/(p+n)$. The best value of $c(n,p)$ of course depends on Ω.

2. Ordinary Poincaré inequality[7] For all function $u \in W^{1,p}(Q(r))$, $1 \le p < \infty$, the inequality

$$\int_{Q(r)} |u - c_u|^p dx \le c r^p \int_{Q(r)} |\nabla u|^p dx \tag{2.2}$$

holds.

3. Young's inequality The following Young's inequality valid for all a,$b > 0$, $\varepsilon > 0$, and $p > 1$

$$ab \le \varepsilon a^{p'} + C(\varepsilon, p) b^p, \qquad \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.3}$$

4. The elementary formula[8] The following formula valid for every $X, Y \in \mathbf{R}^n$ and ε satisfies $0 \le \varepsilon < 1$

$$||X|^{-\varepsilon}X - |Y|^{-\varepsilon}Y| \le 2^{\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}|X - Y|^{1-\varepsilon}. \tag{2.4}$$

Lemma 2.1. *Let $g(x) \in L^r(B_R)$,$1 < r < \infty$,$h(x) \in L^r(B_R)$,$t > r$, if the following Hölder's inequality holds:*

$$\begin{aligned} \fint_{B_{R/2}} |g(x)|^r dx &\le \tau \fint_{B_R} |g(x)|^r dx + C \left(\fint_{B_R} |g(x)|^s dx \right)^{r/s} \\ &+ \fint_{B_R} |h(x)|^r dx \end{aligned}$$

where $0 < R < R_0 \le d(x_0, \partial\Omega), \forall x_0 \in \Omega$, $1 \le s < r, 0 \le \tau < 1, \fint_{B_R} f(x)dx = \frac{1}{|B_R|}\int_{B_R} f(x)dx$. Then these exists $r' = r'(\tau, p, n, C) > r$, such that $g(x) \in L^{r'}_{loc}(\Omega)$;and that

$$\begin{aligned} &\left(\fint_{B_{R/2}} |g(x)|^{r'} dx \right)^{1/r'} \le \\ &C_1 \left[\left(\fint_{B_R} |g(x)|^r dx \right)^{1/r} + \left(\fint_{B_R} |h(x)|^{r'} dx \right)^{1/r'} \right], \end{aligned} \tag{2.5}$$

here C_1 depends only on n, C, r, τ, R_0. see [9].

3. Proof of the theorems

Proof of Theorem 1 Let u be a very weak solution to the nonhomogeneous $K^r_{\psi,\theta}$-obstacle problem. Let

$$E(v, u) = |\bigtriangledown (v - u)|^{r-p} \bigtriangledown (v - u) + |\bigtriangledown u|^{r-p} \bigtriangledown u, \tag{3.1}$$

for every $v \in K^r_{\psi,\theta}(\Omega)$. By the elementary formula (13), we can derive that

$$|E(v, u)| \le 2^{p-r} \frac{p - r + 1}{r - p + 1} |\bigtriangledown v|^{r-p+1}, \tag{3.2}$$

then from (15) we get that

$$\begin{aligned} &\int_\Omega \langle A(x, \bigtriangledown u), |\bigtriangledown u|^{r-p} \bigtriangledown u\rangle dx = \int_\Omega \langle A(x, \bigtriangledown u), E(v, u)\rangle dx \\ &- \int_\Omega \langle A(x, \bigtriangledown u), |\bigtriangledown (v - u)|^{r-p} \bigtriangledown (v - u)\rangle dx. \end{aligned} \tag{3.3}$$

Combining the definition (7), the assumptions (i),(ii) and the estimates (4),(5) and (16), we obtain

$$\begin{aligned} &\alpha \int_\Omega |\bigtriangledown u|^r dx \le \int_\Omega \langle A(x, \bigtriangledown u), |\bigtriangledown u|^{r-p} \bigtriangledown u\rangle dx \\ \le\ &\beta 2^{p-r} \frac{p - r + 1}{r - p + 1} \| \bigtriangledown u\|_r^{p-1} \| \bigtriangledown v\|_r^{r-p+1} \\ &+ C\beta(p - r)\| \bigtriangledown u\|_r^{p-1} (\| \bigtriangledown u\|_r^{r-p+1} + \| \bigtriangledown v\|_r^{r-p+1}) \\ &+ C\|F\|_{\frac{r}{p-1}} (\| \bigtriangledown u\|_r^{r-p+1} + \| \bigtriangledown v\|_r^{r-p+1}). \end{aligned}$$

Then we have, by Young's inequality (12), that

$$\begin{aligned} \int_\Omega |\bigtriangledown u|^r dx &\le C\varepsilon \| \bigtriangledown u\|_r^r + C(\varepsilon)\| \bigtriangledown v\|_r^r \\ &+ C(p - r)\| \bigtriangledown u\|_r^r + C(\varepsilon)\|F\|_{\frac{r}{p-1}}^{\frac{r}{p-1}}, \end{aligned}$$

where $C = C(n,p,r,\beta,\alpha), C(\varepsilon) = C(n,p,r,\varepsilon,\beta,\alpha)$. Choosing r_0 satisfied $C(p-r_0)=1$, thus $C(p-r)=t_1<1$ when $r_0<r$. While then, let ε small enough such that $C\varepsilon+t_1=t<1$, we can deduce that

$$\int_\Omega |\bigtriangledown u|^r dx \leq c\left[\int_\Omega |\bigtriangledown v|^r dx + \int_\Omega |F|^{\frac{r}{p-1}} dx\right].$$

where $c = c(n,p,r_0,\alpha,\beta)$. This completes the proof of Theorem 1.

Proof of Theorem 2 Let $Q(2R)\subset\Omega$ be a cube. Fix a cutoff function $\phi\in C_0^\infty(Q(2R))$ such that $0\leq\phi\leq 1$, $|\nabla\phi|\leq\frac{C(n)}{R}$, and $\phi=1$ on $Q(R)$. Consider the function

$$v = u - C_u - \phi^r(u - C_u - (\psi - C_\psi)), \tag{3.4}$$

here C_u and C_ψ denote the mean values of the functions u and ψ, respectively, in $Q(2R)$. We can see,from [5,p162], that $v\in K^r_{\psi-C_u,\theta-C_u}(\Omega)$. Let

$$E_1(v,u) = |X_1|^{-\varepsilon}X_1 + |Y_1|^{-\varepsilon}Y_1, \tag{3.5}$$

here

$$\varepsilon = p-r, X_1 = \phi^r\bigtriangledown u, Y_1 = \bigtriangledown v - \bigtriangledown u.$$

We can obtain, using the elementary formula (4), that

$$\begin{aligned}&|E_1(v,u)|\leq \frac{2^\varepsilon(1+\varepsilon)}{1-\varepsilon}|X_1+Y_1|^{1-\varepsilon}\\ &= \frac{2^{p-r}(p-r+1)}{r-p+1}\times\\ &|\phi^r\bigtriangledown\psi + r\phi^{r-1}\bigtriangledown\phi((\psi-C_\psi)-(u-C_u))|^{r-p+1}.\end{aligned} \tag{3.6}$$

And because $u-C_u$ is a very weak solution to the $K^r_{\psi-C_u,\theta-C_u}$-obstacle problem, we have

$$\begin{aligned}&\int_\Omega\langle A(x,\bigtriangledown u), |\phi^r\bigtriangledown u|^{r-p}\phi^r\bigtriangledown u\rangle dx\\ \leq\ &\int_{Q(2R)}\langle A(x,\bigtriangledown u), E_1(v,u)\rangle dx -\\ &\int_{Q(2R)}\langle A(x,\bigtriangledown u), h_{v,u}\rangle dx - \int_{Q(2R)}\langle F,\bigtriangledown\phi_{v,u}\rangle dx\\ =\ &I_1+I_2+I_3.\end{aligned} \tag{3.7}$$

By the assumption (i), we estimate the left end of (21) first .

$$\begin{aligned}&\int_\Omega\langle A(x,\bigtriangledown u), |\phi^r\bigtriangledown u|^{r-p}\phi^r\bigtriangledown u\rangle dx\\ &\geq\alpha\int_{Q(r)}|\bigtriangledown u|^r dx,\end{aligned} \tag{3.8}$$

Using (ii) , (20) and Hölder's inequality, we obtain

$$
\begin{aligned}
|I_1| &\leq \beta \int_{Q(2R)} |\bigtriangledown u|^{p-1} |E_1(v,u)| dx \\
&\leq \beta \frac{2(p-r+1)}{r-p+1} \left(\int_{Q(2R)} |\bigtriangledown u|^r dx \right)^{\frac{p-1}{r}} \times \\
&\left(\int_{Q(2R)} |\bigtriangledown \psi|^r dx \right)^{\frac{r-p+1}{r}} \\
&+\beta \frac{2(p-r+1)}{r-p+1} \left(\int_{Q(2R)} |\bigtriangledown u|^r dx \right)^{\frac{p-1}{r}} \times \\
&\left(\int_{Q(2R)} |r\phi^{r-1} \bigtriangledown \phi((\psi - C_\psi) - (u - C_u))|^r dx \right)^{\frac{r-p+1}{r}} \qquad (3.9)
\end{aligned}
$$

Similarly, using Hölder's inequality, (ii),(4) and (5),we have

$$
\begin{aligned}
|I_2| &\leq \beta \int_{Q(2R)} |\bigtriangledown u|^{p-1} |h_{v,u}| dx \\
&\leq c\beta(p-r) \left(\int_{Q(2R)} |\bigtriangledown u|^r dx \right)^{\frac{p-1}{r}} \times \\
&\left(\int_{Q(2R)} |\bigtriangledown v - \bigtriangledown u|^r dx \right)^{\frac{r-p+1}{r}}, \qquad (3.10) \\
|I_3| &\leq \int_{Q(2R)} |F| |\bigtriangledown \varphi_{v,u}| dx \\
&\leq c \left(\int_{Q(2R)} |F|^{\frac{r}{p-1}} dx \right)^{\frac{p-1}{r}} \times \\
&\left(\int_{Q(2R)} |\bigtriangledown (v-u)|^r dx \right)^{\frac{r-p+1}{r}}, \qquad (3.11)
\end{aligned}
$$

Pay attention to the following

$$
\bigtriangledown v - \bigtriangledown u = -\phi^r \bigtriangledown u + \phi^r \bigtriangledown \psi + r\phi^{r-1} \bigtriangledown \phi((\psi - C_\psi) - (u - C_u)),
$$

and $C\beta(p-r)=\tau_1<2^{-n}$, here, $r_1<r<p$, r_1 satisfied $C\beta(p-r_1)=2^{-n}$. Then, from (21)-(25),using Young inequality, with $C=C(n,p,r,r_1,\beta,\alpha,\varepsilon)$, we have

$$\begin{aligned}&\int_{Q(R)}|\bigtriangledown u|^r dx\le 2C\varepsilon\int_{Q(2R)}|\bigtriangledown u|^r dx\\&+C\int_{Q(2R)}\left(|\bigtriangledown \psi|^r+|F|^{\frac{r}{p-1}}\right)dx\\&+C\int_{Q(2R)}|\phi^{r-1}\bigtriangledown\phi((\psi-C_\psi)-(u-C_u))|^r dx\\&+\tau_1\int_{Q(2R)}|\bigtriangledown u|^r dx,\end{aligned}\tag{3.12}$$

Let ε small enough so that $2C\varepsilon+\tau_1=\tau_2<2^{-n}$, we obtain from the above inequality

$$\begin{aligned}&\int_{Q(R)}|\bigtriangledown u|^r dx\le \tau_2\int_{Q(2R)}|\bigtriangledown u|^r dx+\\&C\int_{Q(2R)}\left(|\bigtriangledown \psi|^r+|F|^{\frac{r}{p-1}}\right)dx+\\&C\int_{Q(2R)}|\phi^{r-1}\bigtriangledown\phi((\psi-C_\psi)-(u-C_u))|^r dx.\end{aligned}\tag{3.13}$$

Next we estimate the last integral in (27) by using the ordinary Poincaré inequality and the Sobolev-Poincaré inequality, together with $|\bigtriangledown\varphi\le\frac{C(n)}{R}|$, these give

$$\begin{aligned}&\int_{Q(R)}|\bigtriangledown u|^r dx\le \tau_2\int_{Q(2R)}|\bigtriangledown u|^r dx\\&+C\int_{Q(2R)}\left(|\bigtriangledown \psi|^r+|F|^{\frac{r}{p-1}}\right)dx\\&+CR^{-r}\left(\int_{Q(2R)}|\bigtriangledown u|^{\frac{nr}{n+r}}dx\right)^{\frac{n+r}{n}}.\end{aligned}\tag{3.14}$$

This , with $\tau=2^n\tau_2<1$, implies

$$\begin{aligned}&⨍_{Q(R)}|\bigtriangledown u|^r dx\le \tau⨍_{Q(2R)}|\bigtriangledown u|^r dx\\&+C⨍_{Q(2R)}\left(|\bigtriangledown \psi|^r+|F|^{\frac{r}{p-1}}\right)dx\\&+CR^{-r}\left(⨍_{Q(2R)}|\bigtriangledown u|^{\frac{nr}{n+r}}dx\right)^{\frac{n+r}{n}}.\end{aligned}\tag{3.15}$$

This is a inverse Hölder inequality for $\frac{nr}{n+r}<r$. We can obtain from Lemma 1 and the Sobolev imbedding theorem that, there exist $q>r$ so that$u\in W^{1,q}_{loc}(\Omega)$, we have proved Theorem 2.

Proof for Theorem 3 Since Ω is bounded, we can choose a cube $Q_0=Q\,(2R_0)$ such that $\Omega\subset Q\,(R_0)$. Next let $Q\,(2R)\subset Q_0$. There are two possibilities: 1) $Q\left(\frac{3}{2}R\right)\subset\Omega$ or 2) $Q\left(\frac{3}{2}r\right)\bigcap\partial\Omega\ne\emptyset$. In the case 1) we can follow the proof for Theorem 2 to obtain the inverse Hölder inequality with $0<\tau<1$. In the case

2), note that replacing θ by $\theta_1 = max(\theta, \psi)$, we may assume that the boundary function θ satisfies $\theta \geq \psi$ in Ω. Indeed, $\theta_1 = (\psi - \theta)^+ + \theta$ and since

$$0 \leq (\psi - \theta)^+ \leq (u - \theta)^+ \in W_0^{1,r}(\Omega),$$

the function $(\psi - \theta)^+ \in W_0^{1,r}(\Omega)$, and hence $u - \theta_1$ belongs to $W_0^{1,r}(\Omega)$. Next let

$$v = u - \varphi^r(u - \theta),$$

in Ω where $\varphi \in C_0^\infty(Q(2R))$ is a similar cut-off function as in the proof of Theorem 2. Now $v \in K^r_{\psi,\theta}$ because $v - \theta \in W_0^{1,r}(\Omega)$ and $u \geq \psi$, $\theta \geq \psi$ a.e. yields

$$v = (1 - \varphi^r)u + \varphi^r\theta \geq (1 - \varphi^r)\psi + \varphi^r\psi = \psi.$$

a.e. Let

$$E_2(v, u) = |X_2|^{-\varepsilon}X_2 + |Y_2|^{-\varepsilon}Y_2, \tag{3.16}$$

where

$$\varepsilon = p - r, X_2 = \phi^r \bigtriangledown u, Y_2 = \bigtriangledown v - \bigtriangledown u.$$

Then using the elementary formula (13) , the estimates (4) , (5) , the lemma and the same method of [5] , we obtain from the above inequality that $u \in L^\delta(\Omega)$. Setting $q = min(t, \delta) > r$ we see that $u \in W^{1,q}(\Omega)$ in the case $r < n$. If $r \geq n$, then we can apply the above reasoning for any $r^* < \infty$ to conclude that $u \in L^\delta(\Omega)$ and hence $u \in W^{1,q}(\Omega)$ with $q = min(t, s) > r$ in this case. The theorem follows.

Acknowledgment

The authors would like to thank the anonymous referees and the conference group for their very careful reading of the manuscript and fruitful comments and suggestions. This research is partly supported by Natural Science Foundation of Hebei Province (A2011201011),Natural Science Foundation of Hebei Province (A2012201054), and the Youth Foundation of Hebei University(2010Q24).

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