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INVARIANT SUBMANIFOLDS OF SASAKIAN MANIFOLDS ADMITTING QUARTER SYMMETRIC METRIC CONNECTION-II

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Abstract. In the present paper we have studied invariant submanifolds of Sasakian manifolds admitting quarter symmetric metric connection and obtained some interesting results.

1. Quarter symmetric metric connection

The geometry of invariant submanifolds of Sasakian manifolds is carried out from 1970's by M. Kon [7], K. Yano and M. Kon [12]. It is proved that invariant submanifold of Sasakian structure also carries Sasakian structure. The authors [7, 12] had shown that M is totally geodesic if the second fundamental form σ is parallel, 2-parallel, semiparallel and established their equivalence. Further they had shown the ϕ-sectional curvature K of M is less than or equal to the sectional curvature $\widetilde{K}$ of $\widetilde{M}$ and they are equal if and only if M is totally geodesic. Also the authors B.S. Anitha and C.S. Bagewadi [1] have studied and proved that M is totally geodesic when the second fundamental form is recurrent, 2-recurrent, generalized 2-recurrent, 2-semiparallel, 2-pseudoparallel, 2-Ricci-generalized pseudoparallel and established their equivalence. In this paper we extend the results to invariant submanifolds M of Sasakian manifolds admitting quarter symmetric metric connection.

The paper is organized as follows: This section contains basic concepts of quarter symmetric metric connection and also recurrence and generalized 2-recurrence of a covariant tensor of order k. The section 2, deals with some definitions and notions about immersions of a Riemannian manifold, totally geodesic and Gauss and Weingarten equations. The section 3, gives Sasakian structure with an example. The section 4, deals with invariant submanifolds of Sasakian manifolds admitting quarter symmetric metric connection. In the last section, Recurrent Invariant submanifolds of Sasakian manifolds admitting quarter symmetric metric connection are studied and we have given the final results in the forms of Theorems and Corollaries.

We know that a connection ∇ on a manifold M is called a metric connection if there is a Riemannian metric g on M if $\nabla_g = 0$ otherwise it is non-metric. In 1924, Friedman and J.A. Schouten [4] introduced the notion of a semi-symmetric linear

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connection on a differentiable manifold. In 1932, H.A. Hayden [6] introduced the idea of metric connection with torsion on a Riemannian manifold. In 1970, K. Yano [11] studied some curvature tensors and conditions for semi-symmetric connections in Riemannian manifolds. In 1975's S. Golab [5] defined and studied quarter symmetric linear connection on a differentiable manifold. A linear connection $\overline{\nabla}$ in an n-dimensional Riemannian manifold is said to be a quarter symmetric connection [5]if its torsion tensor T is of the form

$$T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = A(Y)KX - A(X)KY, \tag{1.1}$$

where A is a 1-form and K is a tensor field of type $(1,1)$. If a quarter symmetric linear connection $\overline{\nabla}$ satisfies the condition

$$(\overline{\nabla}_X g)(Y,Z) = 0,$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on the manifold M, then $\overline{\nabla}$ is said to be a quarter symmetric metric connection. For a contact metric manifold admitting quarter symmetric connection, we can take $A = \eta$ and $K = \phi$ to write (1.1) in the form:

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{1.2}$$

The relation between Levi-Civita connection ∇ and quarter symmetric metric connection $\overline{\nabla}$ of a contact metric manifold is given by

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{1.3}$$

2. Isometric Immersion

Let $f : (M,g) \to (\widetilde{M},\widetilde{g})$ be an isometric immersion from an n-dimensional Riemannian manifold (M,g) into $(n+d)$-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$, $n \geq 2$, $d \geq 1$. We denote by ∇ and $\widetilde{\nabla}$ as Levi-Civita connection of M^n and $\widetilde{M}^{n+d}$ respectively. Then the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2.1}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.2}$$

for any tangent vector fields X, Y and the normal vector field N on M, where σ, A and $\nabla^{\perp}$ are the second fundamental form, the shape operator and the normal connection respectively. If the second fundamental form σ is identically zero, then the manifold is said to be *totally geodesic*. The second fundamental form σ and A_N are related by

$$\widetilde{g}(\sigma(X,Y),N) = g(A_N X, Y),$$

for tangent vector fields X, Y. The first and second covariant derivatives of the second fundamental form σ are given by

$$(\widetilde{\nabla}_X \sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{2.3}$$

$$\begin{aligned}(\widetilde{\nabla}^2\sigma)(Z,W,X,Y) &= (\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma)(Z,W), \\ &= \nabla^{\perp}_X((\widetilde{\nabla}_Y\sigma)(Z,W)) - (\widetilde{\nabla}_Y\sigma)(\nabla_X Z, W) \\ &\quad -(\widetilde{\nabla}_X\sigma)(Z,\nabla_Y W) - (\widetilde{\nabla}_{\nabla_X Y}\sigma)(Z,W)\end{aligned} \tag{2.4}$$

respectively, where $\widetilde{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M [3]. If $\widetilde{\nabla}\sigma = 0$, then M is said to have *parallel second fundamental form* [3]. We next define endomorphisms $R(X,Y)$ and $X \wedge_B Y$ of $\chi(M)$ by

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\ (X \wedge_B Y)Z &= B(Y,Z)X - B(X,Z)Y \end{aligned} \tag{2.5}$$

respectively, where $X, Y, Z \in \chi(M)$ and B is a symmetric $(0,2)$-tensor.

Now, for a $(0,k)$-tensor field T, $k \geq 1$ and a $(0,2)$-tensor field B on (M,g), we define the tensor $Q(B,T)$ by

$$\begin{aligned} Q(B,T)(X_1,...,X_k;X,Y) &= -(T(X \wedge_B Y)X_1,...,X_k) \\ &\quad - \cdots - T(X_1,...,X_{k-1}(X \wedge_B Y)X_k). \end{aligned} \tag{2.6}$$

Putting into the above formula $T = \overline{\widetilde{\nabla}}\sigma$ and $B = g$, $B = S$, we obtain the tensors $Q(g, \overline{\widetilde{\nabla}}\sigma)$ and $Q(S, \overline{\widetilde{\nabla}}\sigma)$.

3. Sasakian manifolds

An n-dimensional differentiable manifold M is said to have an almost contact structure (ϕ, ξ, η) if it carries a tensor field ϕ of type $(1,1)$, a vector field ξ and 1-form η on M respectively such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0. \tag{3.1}$$

Thus a manifold M equipped with this structure is called an almost contact manifold and is denoted by (M, ϕ, ξ, η). If g is a Riemannian metric on an almost contact manifold M such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{3.2}$$

where X, Y are vector fields defined on M, then M is said to have an almost contact metric structure (ϕ, ξ, η, g) and M with this structure is called an almost contact metric manifold and is denoted by (M, ϕ, ξ, η, g).

If on (M, ϕ, ξ, η, g) the exterior derivative of 1-form η satisfies,

$$\Phi(X,Y) = d\eta(X,Y) = g(X, \phi Y), \tag{3.3}$$

then (ϕ, ξ, η, g) is said to be a contact metric structure and together with manifold M is called contact metric manifold and Φ is a 2-form. The contact metric structure (M, ϕ, ξ, η, g) is said to be normal if

$$[\phi, \phi](X,Y) + 2d\eta \otimes \xi = 0. \tag{3.4}$$

If the contact metric structure is normal, then it is called a Sasakian structure and M is called a Sasakian manifold. Note that an Almost contact metric manifold defines Sasakian structure if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{3.5}$$

$$\nabla_X \xi = -\phi X. \tag{3.6}$$

Example of Sasakian manifold: Consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3\}$, where (x,y,z) are the standard co-ordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$E_1 = \frac{\partial}{\partial x} - 2ye^z \frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y}, \quad E_3 = e^z \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(E_1, E_2) &= g(E_1, E_3) = g(E_2, E_3) = 0, \\ g(E_1, E_1) &= g(E_2, E_2) = g(E_3, E_3) = 1. \end{aligned}$$

The (ϕ, ξ, η) is given by

$$\begin{aligned} \eta &= 2ydx + e^{-z}dz, \quad \xi = E_3 = e^z \frac{\partial}{\partial z}, \\ \phi E_1 &= E_2, \quad \phi E_2 = -E_1, \quad \phi E_3 = 0. \end{aligned}$$

The linearity property of ϕ and g yields

$$\begin{aligned} \eta(E_3) &= 1, \quad \phi^2 U = -U + \eta(U)E_3, \\ g(\phi U, \phi W) &= g(U, W) - \eta(U)\eta(W), \quad g(U, \xi) = \eta(U), \end{aligned}$$

for any vector fields U, W on M. By definition of Lie bracket, we have

$$[E_1, E_2] = 2E_3.$$

The Levi-Civita connection with respect to above metric g be given by Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Then, we have

$$\begin{aligned} &\nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = E_3, \quad \nabla_{E_1} E_3 = -E_2, \\ &\nabla_{E_2} E_1 = -E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_2} E_3 = E_1, \\ &\nabla_{E_3} E_1 = -E_2, \quad \nabla_{E_3} E_2 = E_1, \quad \nabla_{E_3} E_3 = 0. \end{aligned}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3, i.e., $X = a_1E_1 + a_2E_2 + a_3E_3$ and $Y = b_1E_1 + b_2E_2 + b_3E_3$, where a_i and b_j are scalars. Clearly (ϕ, ξ, η, g) and X, Y satisfy equations (3.1), (3.2), (3.5) and (3.6). Thus M is a Sasakian manifold. Further the following relations hold:

$$R(X, Y)Z = \{g(Y, Z)X - g(X, Z)Y\}, \tag{3.7}$$

$$R(X, Y)\xi = \{\eta(Y)X - \eta(X)Y\}, \tag{3.8}$$

$$R(\xi, X)Y = \{g(X, Y)\xi - \eta(Y)X\}, \tag{3.9}$$

$$R(\xi, X)\xi = \{\eta(X)\xi - X\}, \tag{3.10}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{3.11}$$

$$Q\xi = (n-1)\xi, \tag{3.12}$$

for all vector fields, X, Y, Z and where ∇ denotes the operator of covariant differentiation with respect to g, ϕ is a $(1,1)$ tensor field, S is the Ricci tensor of type $(0,2)$ and R is the Riemannian curvature tensor of the manifold.

4. Invariant submanifolds of Sasakian manifolds admitting Quarter symmetric metric connection

If $\widetilde{M}$ is a Sasakian manifold with structure tensors $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$, then we know that its invariant submanifold M has the induced Sasakian structure (ϕ, ξ, η, g).

(1) Let $\overline{\overline{\nabla}}$ denote quarter symmetric metric connection on the original manifold $\widetilde{M}$.

(2) Let $\widetilde{\nabla}$ denote Levi-Civita connection on $\widetilde{M}$ and

(3) ∇ the induced Levi-Civita connection on the submanifold M.

Further let $\overline{\overline{R}}, \widetilde{R}, R$ be Riemannian curvature tensors on $\widetilde{M}$ and M with respect to $\overline{\overline{\nabla}}, \widetilde{\nabla}, \nabla$ and be given by

$$\overline{\overline{R}}(X,Y)Z = \overline{\overline{\nabla}}_X \overline{\overline{\nabla}}_Y Z - \overline{\overline{\nabla}}_Y \overline{\overline{\nabla}}_X Z - \overline{\overline{\nabla}}_{[X,Y]} Z. \tag{4.1}$$

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z. \tag{4.2}$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{4.3}$$

Lemma 4.1. *Let $\widetilde{M}$ be a contact metric manifold which admits quarter symmetric metric connection $\overline{\overline{\nabla}}$ and M an invariant submanifold of $\widetilde{M}$ with a linear connection $\widetilde{\nabla}$. Then (1) M admits quarter symmetric metric connection, (2) the second fundamental forms σ and $\overline{\sigma}$ with respect to $\widetilde{\nabla}$ and $\overline{\overline{\nabla}}$ are equal.*

Proof. We know that the contact metric structure $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ on $\widetilde{M}$ induces (ϕ, ξ, η, g) on invariant submanifold. By virtue of (1.3), we get

$$\overline{\overline{\nabla}}_X Y = \widetilde{\nabla}_X Y - \eta(X)\phi Y. \tag{4.4}$$

By using (2.1) in (4.4), we get

$$\overline{\overline{\nabla}}_X Y = \nabla_X Y + \sigma(X,Y) - \eta(X)\phi Y. \tag{4.5}$$

Now Gauss formula (2.1) corresponding to quarter symmetric metric connection is given by

$$\overline{\overline{\nabla}}_X Y = \overline{\nabla}_X Y + \overline{\sigma}(X,Y). \tag{4.6}$$

Equating (4.5) and (4.6), we get (1.3) and

$$\overline{\sigma}(X,Y) = \sigma(X,Y). \tag{4.7}$$

□

A submanifold M of a Sasakian manifold $\widetilde{M}$ is called an invariant submanifold of $\widetilde{M}$, if for each $x \in M$, $\phi(T_xM) \subset T_xM$. As a consequence, ξ becomes tangent to M. In an invariant submanifold of a Sasakian manifold

$$\sigma(X,\xi) = 0, \tag{4.8}$$

for any vector X tangent to M. Hence by vitrue of (4.7), we get $\overline{\sigma}(X,\xi) = 0$.

Now we introduce the definitions of 2-semiparallel, 2-pseudoparallel and 2-Ricci-generalized pseudoparallel with respect to quarter symmetric metric connection.

definition 4.2. *An immersion is said to be 2-semiparallel, 2-pseudoparallel and 2-Ricci-generalized pseudoparallel with respect to quarter symmetric metric connection, respectively, if the following conditions hold for all vector fields X, Y tangent* *to M*

$$\begin{aligned} \overline{\overline{\widetilde{R}}} \cdot \overline{\overline{\widetilde{\nabla}}}\sigma &= 0, & (4.9)\\ \overline{\overline{\widetilde{R}}} \cdot \overline{\overline{\widetilde{\nabla}}}\sigma &= L_1 Q(S, \overline{\overline{\widetilde{\nabla}}}\sigma) \quad \text{and} & (4.10)\\ \overline{\overline{\widetilde{R}}} \cdot \overline{\overline{\widetilde{\nabla}}}\sigma &= L_2 Q(S, \overline{\overline{\widetilde{\nabla}}}\sigma), & (4.11) \end{aligned}$$

where $\overline{\overline{\widetilde{R}}}$ denotes the curvature tensor with respect to connection $\overline{\overline{\widetilde{\nabla}}}$. Here L_1 and L_2 are functions depending on $\overline{\overline{\widetilde{\nabla}}}\sigma$.

Lemma 4.3. *Let M be an invariant submanifold of Contact manifold $\widetilde{M}$ which admits quarter symmetric metric connection. Then Gauss and Weingarten formulae with respect to quarter symmetric metric connection are given by*

$$\begin{aligned} &tan(\overline{\overline{\widetilde{R}}}(X,Y)Z) = R(X,Y)Z - \eta(X)\phi\nabla_Y Z - \eta(Y)\nabla_X \phi Z \qquad (4.12)\\ &+\eta(Y)\phi\nabla_X Z + \eta(X)\nabla_Y \phi Z + \eta([X,Y])\phi Z + tan\left\{\overline{\overline{\widetilde{\nabla}}}_X \{\sigma(Y,Z)\}\right.\\ &\left.-\overline{\overline{\widetilde{\nabla}}}_Y \{\sigma(X,Z)\} + \overline{\overline{\widetilde{\nabla}}}_Y \eta(X)\phi Z - \overline{\overline{\widetilde{\nabla}}}_X \eta(Y)\phi Z\right\}, \end{aligned}$$

$$\begin{aligned} &nor(\overline{\overline{\widetilde{R}}}(X,Y)Z) = \sigma(X, \nabla_Y Z) - \eta(Y)\sigma(X,\phi Z) - \sigma(Y, \nabla_X Z) \qquad (4.13)\\ &+\eta(X)\sigma(Y,\phi Z) - \sigma([X,Y],Z) + nor\left\{\overline{\overline{\widetilde{\nabla}}}_X \{\sigma(Y,Z)\}\right.\\ &\left.-\overline{\overline{\widetilde{\nabla}}}_Y \{\sigma(X,Z)\} + \overline{\overline{\widetilde{\nabla}}}_Y \eta(X)\phi Z - \overline{\overline{\widetilde{\nabla}}}_X \eta(Y)\phi Z\right\}. \end{aligned}$$

Proof. By using (1.3) and (2.1) in (4.1), we get

$$\begin{aligned} &\overline{\overline{\widetilde{R}}}(X,Y)Z = R(X,Y)Z + \sigma(X, \nabla_Y Z) - \eta(X)\phi\nabla_Y Z \qquad (4.14)\\ &+\overline{\overline{\widetilde{\nabla}}}_X \{\sigma(Y,Z)\} - \overline{\overline{\widetilde{\nabla}}}_X \eta(Y)\phi Z - \eta(Y)\nabla_X\phi Z - \eta(Y)\sigma(X,\phi Z)\\ &-\sigma(Y,\nabla_X Z) + \eta(Y)\phi\nabla_X Z - \overline{\overline{\widetilde{\nabla}}}_Y \{\sigma(X,Z)\} + \overline{\overline{\widetilde{\nabla}}}_Y \eta(X)\phi Z\\ &+\eta(X)\nabla_Y \phi Z + \eta(X)\sigma(Y,\phi Z) - \sigma([X,Y],Z) + \eta([X,Y])\phi Z. \end{aligned}$$

Comparing tangential and normal part of (4.14), we obtain Gauss and Weingarten formulae (4.12) and (4.13). □

We obtain the condition in the following lemma for 2-semi, 2-pseudo and 2-Ricci-generalized pseudoparallelism for invariant submanifold M of Sasakian manifold $\widetilde{M}$.

Lemma 4.4. *Let M be an invariant submanifold of Contact manifold $\widetilde{M}$ which admits quarter symmetric metric connection. Then*

$$
\begin{aligned}
&(\overline{\overline{R}}(X,Y)\cdot\overline{\overline{\nabla}}\sigma)(U,V,W)=\overline{\overline{R}}(X,Y)\left\{\overline{\nabla}^{\perp}_{U}\sigma(V,W)-\sigma(\overline{\nabla}_{U}V,W)\right. \\
&\left.-\sigma(V,\overline{\nabla}_{U}W)\right\}-\overline{\overline{\nabla}}\sigma(R(X,Y)U,V,W)-\overline{\overline{\nabla}}\sigma(U,R(X,Y)V,W) \\
&-\overline{\overline{\nabla}}\sigma(U,V,R(X,Y)W)-\overline{\overline{\nabla}}\sigma(\sigma(X,\nabla_{Y}U),V,W)+\eta(X)\overline{\overline{\nabla}}\sigma(\phi\nabla_{Y}U,V,W) \\
&-\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_{X}\left\{\sigma(Y,U)\right\},V,W)+\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_{X}\eta(Y)\phi U,V,W)+\eta(Y)\overline{\overline{\nabla}}\sigma(\nabla_{X}\phi U,V,W) \\
&+\eta(Y)\overline{\overline{\nabla}}\sigma(\sigma(X,\phi U),V,W)+\overline{\overline{\nabla}}\sigma(\sigma(Y,\nabla_{X}U),V,W)-\eta(Y)\overline{\overline{\nabla}}\sigma(\phi\nabla_{X}U,V,W) \\
&+\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_{Y}\left\{\sigma(X,U)\right\},V,W)-\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_{Y}\eta(X)\phi U,V,W)-\eta(X)\overline{\overline{\nabla}}\sigma(\nabla_{Y}\phi U,V,W) \\
&-\eta(X)\overline{\overline{\nabla}}\sigma(\sigma(Y,\phi U),V,W)+\overline{\overline{\nabla}}\sigma(\sigma([X,Y],U),V,W)-\eta([X,Y])\overline{\overline{\nabla}}\sigma(\phi U,V,W) \\
&-\overline{\overline{\nabla}}\sigma(U,\sigma(X,\nabla_{Y}V),W)+\eta(X)\overline{\overline{\nabla}}\sigma(U,\phi\nabla_{Y}V,W)-\overline{\overline{\nabla}}\sigma(U,\overline{\overline{\nabla}}_{X}\left\{\sigma(Y,V)\right\},W) \\
&+\overline{\overline{\nabla}}\sigma(U,\overline{\overline{\nabla}}_{X}\eta(Y)\phi V,W)+\eta(Y)\overline{\overline{\nabla}}\sigma(U,\nabla_{X}\phi V,W)+\eta(Y)\overline{\overline{\nabla}}\sigma(U,\sigma(X,\phi V),W) \\
&+\overline{\overline{\nabla}}\sigma(U,\sigma(Y,\nabla_{X}V),W)-\eta(Y)\overline{\overline{\nabla}}\sigma(U,\phi\nabla_{X}V,W)+\overline{\overline{\nabla}}\sigma(U,\overline{\overline{\nabla}}_{Y}\left\{\sigma(X,V)\right\},W) \\
&-\overline{\overline{\nabla}}\sigma(U,\overline{\overline{\nabla}}_{Y}\eta(X)\phi V,W)-\eta(X)\overline{\overline{\nabla}}\sigma(U,\nabla_{Y}\phi V,W)-\eta(X)\overline{\overline{\nabla}}\sigma(U,\sigma(Y,\phi V),W) \\
&+\overline{\overline{\nabla}}\sigma(U,\sigma([X,Y],V),W)+\eta([X,Y])\overline{\overline{\nabla}}\sigma(U,\phi V,W)-\overline{\overline{\nabla}}\sigma(U,V,\sigma(X,\nabla_{Y}W)) \\
&+\eta(X)\overline{\overline{\nabla}}\sigma(U,V,\phi\nabla_{Y}W)-\overline{\overline{\nabla}}\sigma(U,V,\overline{\overline{\nabla}}_{X}\left\{\sigma(Y,W)\right\})+\overline{\overline{\nabla}}\sigma(U,V,\overline{\overline{\nabla}}_{X}\eta(Y)\phi W) \\
&+\eta(Y)\overline{\overline{\nabla}}\sigma(U,V,\nabla_{X}\phi W)+\eta(Y)\overline{\overline{\nabla}}\sigma(U,V,\sigma(X,\phi W))+\overline{\overline{\nabla}}\sigma(U,V,\sigma(Y,\nabla_{X}W)) \\
&-\eta(Y)\overline{\overline{\nabla}}\sigma(U,V,\phi\nabla_{X}W)+\overline{\overline{\nabla}}\sigma(U,V,\overline{\overline{\nabla}}_{Y}\left\{\sigma(X,W)\right\})-\overline{\overline{\nabla}}\sigma(U,V,\overline{\overline{\nabla}}_{Y}\eta(X)\phi W) \\
&-\eta(X)\overline{\overline{\nabla}}\sigma(U,V,\nabla_{Y}\phi W)-\eta(X)\overline{\overline{\nabla}}\sigma(U,V,\sigma(Y,\phi W))+\overline{\overline{\nabla}}\sigma(U,V,\sigma([X,Y],W)) \\
&-\eta([X,Y])\overline{\overline{\nabla}}\sigma(U,V,\phi W),
\end{aligned}
\tag{4.15}
$$

for all vector fields X, Y, U, V and W tangent to M, where

$$R^{\perp}(X,Y)=[\nabla^{\perp}_{X},\nabla^{\perp}_{Y}]-\nabla^{\perp}_{[X,Y]}.$$

Proof. We know, from tensor algebra, that

$$
\begin{aligned}
&(\overline{\overline{R}}(X,Y)\overline{\overline{\nabla}}\sigma)(U,V,W)=\overline{\overline{R}}(X,Y)\overline{\overline{\nabla}}\sigma(U,V,W) \\
&-\overline{\overline{\nabla}}\sigma(\overline{\overline{R}}(X,Y)U,V,W)-\overline{\overline{\nabla}}\sigma(U,\overline{\overline{R}}(X,Y)V,W) \\
&-\overline{\overline{\nabla}}\sigma(U,V,\overline{\overline{R}}(X,Y)W).
\end{aligned}
\tag{4.16}
$$

We write the equation (2.3) with respect to quarter symmetric metric connection and in the form we have the following equalities:

$$\overline{\overline{\nabla}}\sigma(U,V,W)=\overline{\nabla}^{\perp}_{U}\sigma(V,W)-\sigma(\overline{\nabla}_{U}V,W)-\sigma(V,\overline{\nabla}_{U}W). \tag{4.17}$$

By using (4.14) in $\overline{\widetilde{\nabla}}\sigma(\overline{\widetilde{R}}(X,Y)U,V,W)$, $\overline{\widetilde{\nabla}}\sigma(U,\overline{\widetilde{R}}(X,Y)V,W)$ and $\overline{\widetilde{\nabla}}\sigma(U,V,\overline{\widetilde{R}}(X,Y)W)$ to get

$$\begin{aligned}
&\overline{\widetilde{\nabla}}\sigma(\overline{\widetilde{R}}(X,Y)U,V,W) = \overline{\widetilde{\nabla}}\sigma(R(X,Y)U,V,W)\\
&+\overline{\widetilde{\nabla}}\sigma(\sigma(X,\nabla_Y U),V,W) - \eta(X)\overline{\widetilde{\nabla}}\sigma(\phi\nabla_Y U,V,W)\\
&+\overline{\widetilde{\nabla}}\sigma(\overline{\widetilde{\nabla}}_X\left\{\sigma(Y,U)\right\},V,W) - \overline{\widetilde{\nabla}}\sigma(\overline{\widetilde{\nabla}}_X\eta(Y)\phi U,V,W)\\
&-\eta(Y)\overline{\widetilde{\nabla}}\sigma(\nabla_X\phi U,V,W) - \eta(Y)\overline{\widetilde{\nabla}}\sigma(\sigma(X,\phi U),V,W)\\
&-\overline{\widetilde{\nabla}}\sigma(\sigma(Y,\nabla_X U),V,W) + \eta(Y)\overline{\widetilde{\nabla}}\sigma(\phi\nabla_X U,V,W)\\
&-\overline{\widetilde{\nabla}}\sigma(\overline{\widetilde{\nabla}}_Y\left\{\sigma(X,U)\right\},V,W) + \overline{\widetilde{\nabla}}\sigma(\overline{\widetilde{\nabla}}_Y\eta(X)\phi U,V,W)\\
&+\eta(X)\overline{\widetilde{\nabla}}\sigma(\nabla_Y\phi U,V,W) + \eta(X)\overline{\widetilde{\nabla}}\sigma(\sigma(Y,\phi U),V,W)\\
&-\overline{\widetilde{\nabla}}\sigma(\sigma([X,Y],U),V,W) + \eta([X,Y])\overline{\widetilde{\nabla}}\sigma(\phi U,V,W),
\end{aligned}\tag{4.18}$$

$$\begin{aligned}
&\overline{\widetilde{\nabla}}\sigma(U,\overline{\widetilde{R}}(X,Y)V,W) = \overline{\widetilde{\nabla}}\sigma(U,R(X,Y)V,W)\\
&+\overline{\widetilde{\nabla}}\sigma(U,\sigma(X,\nabla_Y V),W) - \eta(X)\overline{\widetilde{\nabla}}\sigma(U,\phi\nabla_Y V,W)\\
&+\overline{\widetilde{\nabla}}\sigma(U,\overline{\widetilde{\nabla}}_X\left\{\sigma(Y,V)\right\},W) - \overline{\widetilde{\nabla}}\sigma(U,\overline{\widetilde{\nabla}}_X\eta(Y)\phi V,W)\\
&-\eta(Y)\overline{\widetilde{\nabla}}\sigma(U,\nabla_X\phi V,W) - \eta(Y)\overline{\widetilde{\nabla}}\sigma(U,\sigma(X,\phi V),W)\\
&-\overline{\widetilde{\nabla}}\sigma(U,\sigma(Y,\nabla_X V),W) + \eta(Y)\overline{\widetilde{\nabla}}\sigma(U,\phi\nabla_X V,W)\\
&-\overline{\widetilde{\nabla}}\sigma(U,\overline{\widetilde{\nabla}}_Y\left\{\sigma(X,V)\right\},W) + \overline{\widetilde{\nabla}}\sigma(U,\overline{\widetilde{\nabla}}_Y\eta(X)\phi V,W)\\
&+\eta(X)\overline{\widetilde{\nabla}}\sigma(U,\nabla_Y\phi V,W) + \eta(X)\overline{\widetilde{\nabla}}\sigma(U,\sigma(Y,\phi V),W)\\
&-\overline{\widetilde{\nabla}}\sigma(U,\sigma([X,Y],V),W) + \eta([X,Y])\overline{\widetilde{\nabla}}\sigma(U,\phi V,W)
\end{aligned}\tag{4.19}$$

and

$$\begin{aligned}
&\overline{\widetilde{\nabla}}\sigma(U,V,\overline{\widetilde{R}}(X,Y)W) = \overline{\widetilde{\nabla}}\sigma(U,V,R(X,Y)W)\\
&+\overline{\widetilde{\nabla}}\sigma(U,V,\sigma(X,\nabla_Y W)) - \eta(X)\overline{\widetilde{\nabla}}\sigma(U,V,\phi\nabla_Y W)\\
&+\overline{\widetilde{\nabla}}\sigma(U,V,\overline{\widetilde{\nabla}}_X\left\{\sigma(Y,W)\right\}) - \overline{\widetilde{\nabla}}\sigma(U,V,\overline{\widetilde{\nabla}}_X\eta(Y)\phi W)\\
&-\eta(Y)\overline{\widetilde{\nabla}}\sigma(U,V,\nabla_X\phi W) - \eta(Y)\overline{\widetilde{\nabla}}\sigma(U,V,\sigma(X,\phi W))\\
&-\overline{\widetilde{\nabla}}\sigma(U,V,\sigma(Y,\nabla_X W)) + \eta(Y)\overline{\widetilde{\nabla}}\sigma(U,V,\phi\nabla_X W)\\
&-\overline{\widetilde{\nabla}}\sigma(U,V,\overline{\widetilde{\nabla}}_Y\left\{\sigma(X,W)\right\}) + \overline{\widetilde{\nabla}}\sigma(U,V,\overline{\widetilde{\nabla}}_Y\eta(X)\phi W)\\
&+\eta(X)\overline{\widetilde{\nabla}}\sigma(U,V,\nabla_Y\phi W) + \eta(X)\overline{\widetilde{\nabla}}\sigma(U,V,\sigma(Y,\phi W))\\
&-\overline{\widetilde{\nabla}}\sigma(U,V,\sigma([X,Y],W)) + \eta([X,Y])\overline{\widetilde{\nabla}}\sigma(U,V,\phi W).
\end{aligned}\tag{4.20}$$

Substituting (4.17) – (4.20) into (4.16), we get (4.15). □

5. 2-semiparallel, 2-pseudoparallel and 2-Ricci-generalized pseudoparallel Invariant submanifolds of Sasakian manifolds admitting Quarter symmetric metric connection

We consider invariant submanifolds of Sasakian manifolds admitting quarter symmetric metric connection satisfying the conditions $\overline{\widetilde{R}} \cdot \overline{\widetilde{\nabla}}\sigma = 0$, $\overline{\widetilde{R}} \cdot \overline{\widetilde{\nabla}}\sigma = L_1 Q(S, \overline{\widetilde{\nabla}}\sigma)$, $\overline{\widetilde{R}} \cdot \overline{\widetilde{\nabla}}\sigma = L_2 Q(S, \overline{\widetilde{\nabla}}\sigma)$ and prove the following theorems

Theorem 5.1. *Let M be an invariant submanifold of a Sasakian manifold with a quarter symmetric metric connection $\widetilde{M}$. Then M is 2-semiparallel with respect to quarter symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let M be 2-semiparallel satisfying $\overline{\widetilde{R}} \cdot \overline{\widetilde{\nabla}}\sigma = 0$. Put $X = V = \xi$ and use (3.1), (3.6) and (4.8) in (4.15) to get

$$\begin{aligned}
0 = &-\overline{\widetilde{R}}(\xi, Y)\left\{\sigma(\overline{\nabla}_U \xi, W) + \sigma(\xi, \overline{\nabla}_U W)\right\} - \overline{\widetilde{\nabla}}\sigma(R(\xi, Y)U, \xi, W) \\
&-\overline{\widetilde{\nabla}}\sigma(U, R(\xi, Y)\xi, W) - \overline{\widetilde{\nabla}}\sigma(U, \xi, R(\xi, Y)W) + \overline{\widetilde{\nabla}}\sigma(\phi\nabla_Y U, \xi, W) \\
&-\overline{\widetilde{\nabla}}\sigma(\overline{\widetilde{\nabla}}_\xi \sigma(Y, U), \xi, W) + \overline{\widetilde{\nabla}}\sigma(\overline{\widetilde{\nabla}}_\xi \eta(Y)\phi U, \xi, W) + \eta(Y)\overline{\widetilde{\nabla}}\sigma(\nabla_\xi \phi U, \xi, W) \\
&+\overline{\widetilde{\nabla}}\sigma(\sigma(Y, \nabla_\xi U), \xi, W) - \eta(Y)\overline{\widetilde{\nabla}}\sigma(\phi\nabla_\xi W, \xi, W) - \overline{\widetilde{\nabla}}\sigma(\overline{\widetilde{\nabla}}_Y \phi U, \xi, W) \\
&-\overline{\widetilde{\nabla}}\sigma(\nabla_Y \phi U, \xi, W) - \overline{\widetilde{\nabla}}\sigma(\sigma(Y, \phi U), \xi, W) + \overline{\widetilde{\nabla}}\sigma(\sigma([\xi, Y], U), \xi, W) \\
&-\eta([\xi, Y])\overline{\widetilde{\nabla}}\sigma(\phi U, \xi, W) + \overline{\widetilde{\nabla}}\sigma(U, \phi\nabla_Y \xi, W) + \overline{\widetilde{\nabla}}\sigma(U, \xi, \phi\nabla_Y W) \\
&-\overline{\widetilde{\nabla}}\sigma(U, \xi, \overline{\widetilde{\nabla}}_\xi \sigma(Y, W)) + \overline{\widetilde{\nabla}}\sigma(U, \xi, \overline{\widetilde{\nabla}}_\xi \eta(Y)\phi W) + \eta(Y)\overline{\widetilde{\nabla}}\sigma(U, \xi, \overline{\widetilde{\nabla}}_\xi \phi W) \\
&+\overline{\widetilde{\nabla}}\sigma(U, \xi, \sigma(Y, \nabla_\xi W)) - \eta(Y)\overline{\widetilde{\nabla}}\sigma(U, \xi, \phi\nabla_\xi W) - \overline{\widetilde{\nabla}}\sigma(U, \xi, \overline{\widetilde{\nabla}}_Y \phi W) \\
&-\overline{\widetilde{\nabla}}\sigma(U, \xi, \nabla_Y \phi W) - \overline{\widetilde{\nabla}}\sigma(U, \xi, \sigma(Y, \phi W)) + \overline{\widetilde{\nabla}}\sigma(U, \xi, \sigma([\xi, Y], W)) \\
&-\eta([\xi, Y])\overline{\widetilde{\nabla}}\sigma(U, \xi, \phi W).
\end{aligned} \tag{5.1}$$

In view of (1.3), (4.8), (3.1), (3.6), (3.9), (3.10) and (4.17) we have the following equalities:

$$\begin{aligned}
&(\overline{\widetilde{\nabla}}(R(\xi, Y)U, \xi, W) = (\overline{\widetilde{\nabla}}_{R(\xi, Y)U}\sigma)(\xi, W), \\
&= \overline{\nabla}^{\perp}_{R(\xi, Y)U}\sigma(\xi, W) - \sigma(\overline{\nabla}_{R(\xi, Y)U}\xi, W) - \sigma(\xi, \overline{\nabla}_{R(\xi, Y)U}W), \\
&= -\eta(U)\sigma(\phi Y, W),
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
&(\overline{\widetilde{\nabla}}\sigma)(U, R(\xi, Y)\xi, W) = (\overline{\widetilde{\nabla}}_U \sigma)(R(\xi, Y)\xi, W), \\
&= \overline{\nabla}^{\perp}_U \sigma(R(\xi, Y)\xi, W) - \sigma(\overline{\nabla}_U R(\xi, Y)\xi, W) - \sigma(R(\xi, Y)\xi, \overline{\nabla}_U W), \\
&= \overline{\nabla}^{\perp}_U \sigma(\{\eta(Y)\xi - Y\}, W) - \sigma(\overline{\nabla}_U \{\eta(Y)\xi - Y\}, W) \\
&+\sigma(Y, \overline{\nabla}_U W)
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&(\overline{\widetilde{\nabla}}\sigma)(U, \xi, R(\xi, Y)W) = (\overline{\widetilde{\nabla}}_U \sigma)(\xi, R(\xi, Y)W), \\
&= \overline{\nabla}^{\perp}_U \sigma(\xi, R(\xi, Y)W) - \sigma(\overline{\nabla}_U \xi, R(\xi, Y)W) - \sigma(\xi, \overline{\nabla}_U R(\xi, Y)W), \\
&= -\eta(W)\sigma(\phi U, Y).
\end{aligned} \tag{5.4}$$

Substituting $(5.2-5.4)$ into (5.1) and $W=\xi$, using (3.1), (3.6), (4.8), we get

$$\begin{aligned}0 &= -\overline{\overline{R}}(\xi,Y)\left\{\sigma(\overline{\nabla}_U\xi,\xi)+\sigma(\xi,\overline{\nabla}_U\xi)\right\}-\overline{\nabla}^{\perp}_U\sigma(\{\eta(Y)\xi-Y\},\xi) \\ &+\sigma(\overline{\nabla}_U\{\eta(Y)\xi-Y\},\xi)-\sigma(Y,\overline{\nabla}_U\xi)+\sigma(\phi U,Y)+\overline{\overline{\widetilde{\nabla}}}\sigma(\phi\nabla_Y U,\xi,\xi) \\ &-\overline{\overline{\widetilde{\nabla}}}\sigma(\overline{\overline{\widetilde{\nabla}}}_\xi\sigma(Y,U),\xi,\xi)+\overline{\overline{\widetilde{\nabla}}}\sigma(\overline{\overline{\widetilde{\nabla}}}_\xi\eta(Y)\phi U,\xi,\xi)+\eta(Y)\overline{\overline{\widetilde{\nabla}}}\sigma(\nabla_\xi\phi U,\xi,\xi) \\ &+\overline{\overline{\widetilde{\nabla}}}\sigma(\sigma(Y,\nabla_\xi U)\xi,\xi)-\eta(Y)\overline{\overline{\widetilde{\nabla}}}\sigma(\phi\nabla_\xi U,\xi,\xi)-\overline{\overline{\widetilde{\nabla}}}\sigma(\overline{\overline{\widetilde{\nabla}}}_Y\phi U,\xi,\xi) \\ &-\overline{\overline{\widetilde{\nabla}}}\sigma(\nabla_Y\phi U,\xi,\xi)-\overline{\overline{\widetilde{\nabla}}}\sigma(\sigma(Y,\phi U),\xi,\xi)+\overline{\overline{\widetilde{\nabla}}}\sigma(\sigma([\xi,Y],U),\xi,\xi) \\ &-\eta([\xi,Y])\overline{\overline{\widetilde{\nabla}}}\sigma(\phi U,\xi,\xi)+\overline{\overline{\widetilde{\nabla}}}\sigma(U,\phi\nabla_Y\xi,\xi)+\overline{\overline{\widetilde{\nabla}}}\sigma(U,\xi,\phi\nabla_Y\xi).\end{aligned} \tag{5.5}$$

Using (1.3), (2.1), (3.1), (3.6), (3.10), (4.8) and (4.17) in (5.5), we get

$$0=4\sigma(\phi U,Y)-2\sigma(\phi\overline{\overline{\widetilde{\nabla}}}_\xi\sigma(Y,U),\xi). \tag{5.6}$$

By definition σ is a vector valued covariant tensor and so $\sigma(U,Y)$ is a vector. Therefore $\overline{\overline{\widetilde{\nabla}}}_\xi\sigma(Y,U)$ is a vector and hence by (4.8), we have

$$\sigma(\phi\overline{\overline{\widetilde{\nabla}}}_\xi\sigma(Y,U),\xi)=0. \tag{5.7}$$

Then from (5.6), we get

$$0=\sigma(\phi U,Y). \tag{5.8}$$

Replacing U by ϕU and using (3.1), (4.8) in (5.8) to obtain $\sigma(U,Y)=0$, i.e., M is totally geodesic. The converse statement is trivial. □

Theorem 5.2. *Let M be an invariant submanifold of a Sasakian manifold with a quarter symmetric metric connection $\widetilde{M}$. Then M is 2-pseudoparallel with respect to quarter symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let M be 2-pseudoparallel satisfying $\overline{\overline{\widetilde{R}}}\cdot\overline{\overline{\widetilde{\nabla}}}\sigma=L_1Q(S,\overline{\overline{\widetilde{\nabla}}}\sigma)$. Put $X=V=\xi$ and use (3.1), (3.6) and (4.8) in (2.6), (4.15), we get

$$\begin{aligned}&-\overline{\overline{R}}(\xi,Y)\left\{\sigma(\overline{\nabla}_U\xi,W)+\sigma(\xi,\overline{\nabla}_U W)\right\}-\overline{\overline{\widetilde{\nabla}}}\sigma(R(\xi,Y)U,\xi,W) \\ &-\overline{\overline{\widetilde{\nabla}}}\sigma(U,R(\xi,Y)\xi,W)-\overline{\overline{\widetilde{\nabla}}}\sigma(U,\xi,R(\xi,Y)W)+\overline{\overline{\widetilde{\nabla}}}\sigma(\phi\nabla_Y U,\xi,W) \\ &-\overline{\overline{\widetilde{\nabla}}}\sigma(\overline{\overline{\widetilde{\nabla}}}_\xi\sigma(Y,U),\xi,W)+\overline{\overline{\widetilde{\nabla}}}\sigma(\overline{\overline{\widetilde{\nabla}}}_\xi\eta(Y)\phi U,\xi,W)+\eta(Y)\overline{\overline{\widetilde{\nabla}}}\sigma(\overline{\overline{\widetilde{\nabla}}}_\xi\phi U,\xi,W) \\ &+\overline{\overline{\widetilde{\nabla}}}\sigma(\sigma(Y,\nabla_\xi U),\xi,W)-\eta(Y)\overline{\overline{\widetilde{\nabla}}}\sigma(\phi\nabla_\xi W,\xi,W)-\overline{\overline{\widetilde{\nabla}}}\sigma(\overline{\overline{\widetilde{\nabla}}}_Y\phi U,\xi,W)\end{aligned} \tag{5.9}$$

$$
\begin{aligned}
&-\overline{\overline{\nabla}}\sigma(\nabla_Y \phi U, \xi, W) - \overline{\overline{\nabla}}\sigma(\sigma(Y,\phi U),\xi,W) + \overline{\overline{\nabla}}\sigma(\sigma([\xi,Y],U),\xi,W)\\
&-\eta([\xi,Y])\overline{\overline{\nabla}}\sigma(\phi U,\xi,W) + \overline{\overline{\nabla}}\sigma(U,\phi\nabla_Y\xi,W) + \overline{\overline{\nabla}}\sigma(U,\xi,\phi\nabla_Y W)\\
&-\overline{\overline{\nabla}}\sigma(U,\xi,\overline{\overline{\nabla}}_\xi\sigma(Y,W)) + \overline{\overline{\nabla}}\sigma(U,\xi,\overline{\overline{\nabla}}_\xi\eta(Y)\phi W) + \eta(Y)\overline{\overline{\nabla}}\sigma(U,\xi,\overline{\overline{\nabla}}_\xi\phi W)\\
&+\overline{\overline{\nabla}}\sigma(U,\xi,\sigma(Y,\nabla_\xi W)) - \eta(Y)\overline{\overline{\nabla}}\sigma(U,\xi,\phi\nabla_\xi W) - \overline{\overline{\nabla}}\sigma(U,\xi,\overline{\overline{\nabla}}_Y\phi W)\\
&-\overline{\overline{\nabla}}\sigma(U,\xi,\nabla_Y\phi W) - \overline{\overline{\nabla}}\sigma(U,\xi,\sigma(Y,\phi W)) + \overline{\overline{\nabla}}\sigma(U,\xi,\sigma([\xi,Y],W))\\
&-\eta([\xi,Y])\overline{\overline{\nabla}}\sigma(U,\xi,\phi W) = -L_1\Big[\eta(W)\Big\{\overline{\nabla}^\perp_\xi\sigma(Y,U) - \sigma(\overline{\nabla}_\xi Y,U) - \sigma(Y,\overline{\nabla}_\xi U)\Big\}\\
&-\overline{\nabla}^\perp_W\sigma(Y,U) + \sigma(\overline{\nabla}_W Y,U) + \sigma(Y,\overline{\nabla}_W U) + g(W,Y)\left\{-\sigma(\overline{\nabla}_\xi\xi,U) - \sigma(\xi,\overline{\nabla}_\xi U)\right\}\\
&-\eta(Y)\Big\{\overline{\nabla}^\perp_\xi\sigma(W,U) - \sigma(\overline{\nabla}_\xi W,U) - \sigma(W,\overline{\nabla}_\xi U)\Big\} + g(W,U)\left\{-\sigma(\overline{\nabla}_\xi Y,\xi)\right.\\
&\left.-\sigma(Y,\overline{\nabla}_\xi\xi)\right\} - \eta(U)\Big\{\overline{\nabla}^\perp_\xi\sigma(Y,W) - \sigma(\overline{\nabla}_\xi Y,W) - \sigma(Y,\overline{\nabla}_\xi W)\Big\}\Big].
\end{aligned}
$$

Substituting (5.2 – 5.4) into (5.9) and $W=\xi$, using (3.1), (3.6), (4.8), we get

$$
\begin{aligned}
0 &= -\overline{\overline{R}}(\xi,Y)\left\{\sigma(\overline{\nabla}_U\xi,\xi) + \sigma(\xi,\overline{\nabla}_U\xi)\right\} - \overline{\nabla}^\perp_U\sigma(\{\eta(Y)\xi - Y\},\xi) \qquad (5.10)\\
&+\sigma(\overline{\nabla}_U\{\eta(Y)\xi - Y\},\xi) - \sigma(Y,\overline{\nabla}_U\xi) + \sigma(\phi U,Y) + \overline{\overline{\nabla}}\sigma(\phi\nabla_Y U,\xi,\xi)\\
&-\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_\xi\sigma(Y,U),\xi,\xi) + \overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_\xi\eta(Y)\phi U,\xi,\xi) + \eta(Y)\overline{\overline{\nabla}}\sigma(\nabla_\xi\phi U,\xi,\xi)\\
&+\overline{\overline{\nabla}}\sigma(\sigma(Y,\nabla_\xi U)\xi,\xi) - \eta(Y)\overline{\overline{\nabla}}\sigma(\phi\nabla_\xi U,\xi,\xi) - \overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_Y\phi U,\xi,\xi)\\
&-\overline{\overline{\nabla}}\sigma(\nabla_Y\phi U,\xi,\xi) - \overline{\overline{\nabla}}\sigma(\sigma(Y,\phi U),\xi,\xi) + \overline{\overline{\nabla}}\sigma(\sigma([\xi,Y],U),\xi,\xi)\\
&-\eta([\xi,Y])\overline{\overline{\nabla}}\sigma(\phi U,\xi,\xi) + \overline{\overline{\nabla}}\sigma(U,\phi\nabla_Y\xi,\xi) + \overline{\overline{\nabla}}\sigma(U,\xi,\phi\nabla_Y\xi).
\end{aligned}
$$

Using (1.3), (2.1), (2.2), (3.1), (3.6), (3.10), (4.8) and (4.17) in (5.10), we get

$$0 = 4\sigma(\phi U,Y) - 2\sigma(\phi\overline{\overline{\nabla}}_\xi\sigma(Y,U),\xi). \qquad (5.11)$$

Now by using (5.7) in (5.11), we get

$$0 = \sigma(\phi U,Y). \qquad (5.12)$$

Replacing U by ϕU and using (3.1), (4.8) in (5.12) to obtain $\sigma(U,Y)=0$, i.e., M is totally geodesic. The converse statement is trivial. □

Theorem 5.3. *Let M be an invariant submanifold of a Sasakian manifold with a quarter symmetric metric connection $\widetilde{M}$. Then M is 2-Ricci-generalized pseudoparallel with respect to quarter symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let M be 2-Ricci-generalized pseudoparallel satisfying $\overline{\overline{R}}\cdot\overline{\overline{\nabla}}\sigma = L_2 Q(S,\overline{\overline{\nabla}}\sigma)$. Put $X=V=\xi$ and use (3.1), (3.6), (3.11) and (4.8) in (2.6), (4.15), we get

$$
\begin{aligned}
&-\overline{\overline{R}}(\xi,Y)\left\{\sigma(\overline{\nabla}_U\xi,W)+\sigma(\xi,\overline{\nabla}_UW)\right\}-\overline{\overline{\nabla}}\sigma(R(\xi,Y)U,\xi,W) \qquad (5.13)\\
&-\overline{\overline{\nabla}}\sigma(U,R(\xi,Y)\xi,W)-\overline{\overline{\nabla}}\sigma(U,\xi,R(\xi,Y)W)+\overline{\overline{\nabla}}\sigma(\phi\nabla_YU,\xi,W)\\
&-\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_\xi\sigma(Y,U),\xi,\xi)+\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_\xi\eta(Y)\phi U,\xi,W)+\eta(Y)\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_\xi\phi U,\xi,W)\\
&+\overline{\overline{\nabla}}\sigma(\sigma(Y,\nabla_\xi U),\xi,W)-\eta(Y)\overline{\overline{\nabla}}\sigma(\phi\nabla_\xi W,\xi,W)-\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_Y\phi U,\xi,W)\\
&-\overline{\overline{\nabla}}\sigma(\nabla_Y\phi U,\xi,W)-\overline{\overline{\nabla}}\sigma(\sigma(Y,\phi U),\xi,W)+\overline{\overline{\nabla}}\sigma(\sigma([\xi,Y],U),\xi,W)\\
&-\eta([\xi,Y])\overline{\overline{\nabla}}\sigma(\phi U,\xi,W)+\overline{\overline{\nabla}}\sigma(U,\phi\nabla_Y\xi,W)+\overline{\overline{\nabla}}\sigma(U,\xi,\phi\nabla_YW)\\
&-\overline{\overline{\nabla}}\sigma(U,\xi,\overline{\overline{\nabla}}_\xi\sigma(Y,W))+\overline{\overline{\nabla}}\sigma(U,\xi,\overline{\overline{\nabla}}_\xi\eta(Y)\phi W)+\eta(Y)\overline{\overline{\nabla}}\sigma(U,\xi,\overline{\overline{\nabla}}_\xi\phi W)\\
&+\overline{\overline{\nabla}}\sigma(U,\xi,\sigma(Y,\nabla_\xi W))-\eta(Y)\overline{\overline{\nabla}}\sigma(U,\xi,\phi\nabla_\xi W)-\overline{\overline{\nabla}}\sigma(U,\xi,\overline{\overline{\nabla}}_Y\phi W)\\
&-\overline{\overline{\nabla}}\sigma(U,\xi,\nabla_Y\phi W)-\overline{\overline{\nabla}}\sigma(U,\xi,\sigma(Y,\phi W))+\overline{\overline{\nabla}}\sigma(U,\xi,\sigma([\xi,Y],W))\\
&-\eta([\xi,Y])\overline{\overline{\nabla}}\sigma(U,\xi,\phi W)=-L_2\Big[(n-1)\eta(W)\Big\{\overline{\nabla}_\xi^\perp\sigma(Y,U)\\
&-\sigma(\overline{\nabla}_\xi Y,U)-\sigma(Y,\overline{\nabla}_\xi U)\Big\}-(n-1)\Big\{\overline{\nabla}_W^\perp\sigma(Y,U)-\sigma(\overline{\nabla}_WY,U)\\
&-\sigma(Y,\overline{\nabla}_WU)\Big\}+S(W,Y)\Big\{\overline{\nabla}_\xi^\perp\sigma(\xi,U)-\sigma(\overline{\nabla}_\xi\xi,U)-\sigma(\xi,\overline{\nabla}_\xi U)\Big\}\\
&+S(W,U)\Big\{\overline{\nabla}_\xi^\perp\sigma(Y,\xi)-\sigma(\overline{\nabla}_\xi Y,\xi)-\sigma(U,\overline{\nabla}_\xi\xi)\ \Big\}\\
&-(n-1)\eta(Y)\Big\{\overline{\nabla}_\xi^\perp\sigma(W,U)-\sigma(\overline{\nabla}_\xi W,U)-\sigma(W,\overline{\nabla}_\xi U)\Big\}\\
&-(n-1)\eta(U)\Big\{\overline{\nabla}_\xi^\perp\sigma(Y,W)-\sigma(\overline{\nabla}_\xi Y,W)-\sigma(Y,\overline{\nabla}_\xi W)\Big\}\Big].
\end{aligned}
$$

Substituting (5.2 – 5.4) into (5.13) and $W=\xi$, using (3.1), (3.6), (4.8), we get

$$
\begin{aligned}
&0=-\overline{\overline{R}}(\xi,Y)\left\{\sigma(\overline{\nabla}_U\xi,\xi)+\sigma(\xi,\overline{\nabla}_U\xi)\right\}-\overline{\nabla}_U^\perp\sigma(\{\eta(Y)\xi-Y\},\xi) \qquad (5.14)\\
&+\sigma(\overline{\nabla}_U\{\eta(Y)\xi-Y\},\xi)-\sigma(Y,\overline{\nabla}_U\xi)+\sigma(\phi U,Y)+\overline{\overline{\nabla}}\sigma(\phi\nabla_YU,\xi,\xi)\\
&-\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_\xi\sigma(Y,U),\xi,\xi)+\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_\xi\eta(Y)\phi U,\xi,\xi)+\eta(Y)\overline{\overline{\nabla}}\sigma(\nabla_\xi\phi U,\xi,\xi)\\
&+\overline{\overline{\nabla}}\sigma(\sigma(Y,\nabla_\xi U)\xi,\xi)-\eta(Y)\overline{\overline{\nabla}}\sigma(\phi\nabla_\xi U,\xi,\xi)-\overline{\overline{\nabla}}\sigma(\overline{\overline{\nabla}}_Y\phi U,\xi,\xi)\\
&-\overline{\overline{\nabla}}\sigma(\nabla_Y\phi U,\xi,\xi)-\overline{\overline{\nabla}}\sigma(\sigma(Y,\phi U),\xi,\xi)+\overline{\overline{\nabla}}\sigma(\sigma([\xi,Y],U),\xi,\xi)\\
&-\eta([\xi,Y])\overline{\overline{\nabla}}\sigma(\phi U,\xi,\xi)+\overline{\overline{\nabla}}\sigma(U,\phi\nabla_Y\xi,\xi)+\overline{\overline{\nabla}}\sigma(U,\xi,\phi\nabla_Y\xi).
\end{aligned}
$$

Using (1.3), (2.1), (2.2), (3.1), (3.6), (3.10), (4.8) and (4.17) in (5.14), we get

$$0=4\sigma(\phi U,Y)-2\sigma(\phi\overline{\overline{\nabla}}_\xi\sigma(Y,U),\xi). \qquad (5.15)$$

Now by using (5.7) in (5.15), we get

$$0=\sigma(\phi U,Y). \qquad (5.16)$$

Replacing U by ϕU and using (3.1), (4.8) in (5.16) to obtain $\sigma(U,Y)=0$, i.e., M is totally geodesic. The converse statement is trivial. □

Using Theorems 5.1 to 5.3, we have the following result

Corollary 5.4. *Let M be an invariant submanifold of a Sasakian manifold with a quarter symmetric metric connection $\widetilde{M}$. Then the following statements are equivalent.*

(1) *M is 2-semiparallel with respect to quarter symmetric metric connection;*
(2) *M is 2-pseudoparallel with respect to quarter symmetric metric connection;*
(3) *M is 2-Ricci-generalized pseudoparallel with respect to quarter symmetric metric connection;*
(4) *M is totally geodesic with respect to Levi-Civita connection.*

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