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EUROPEAN OPTION PRICING UNDER THE REGIME-SWITCHING GARCH MODEL

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Abstract. GARCH model and its extension is the most common and widespread approach used in modeling volatility. However, the existing literature indicates that these models are not able to capture structural changes in the variance process. The financial markets are known to exhibit structural breaks which may be caused by the changing variance process. Therefore, a model that allows its parameters to change with time is necessary so as to account for the structural variance changes. This study proposes a regime switching GARCH model that is able to capture the changing variances in each regime. The model is then utilized to estimate the European option prices for the Russell 2000, Facebook and Google indices with an aim to compare its performance with that of Black-Scholes and Regime switching models. The results indicate that regime switching GARCH perform better than Black-Scholes and Regime switching models when applied to long-dated options contract. However, Black-Scholes model is better for analyzing short-dated options contract.

1. Introduction

The past theoretical and empirical research largely dealt on modeling the financial markets' volatility. According to Bauwens et al.[2], the risk associated with stock returns is commonly measured using volatility. Furthermore, volatility is paramount to managers of Portfolio, option traders and it is useful in risk management, pricing of derivatives, etc. The GARCH model and its extensions are the most frequent and widely used approach to model volatility. The model considers the clustering of volatility and excess kurtosis of the financial data. According to the available empirical evidence, financial market volatility is characterized by persistence, which standard GARCH models are unable to fully address. Dieobold [4] and Lamoureux [13] believe that structural changes in the variance process are to blame for the significant persistence in conditional volatility in these models. Furthermore, because GARCH models do not account for structural changes in the variance process, models that enable the parameters to change over time may be

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more appropriate for volatility modeling. In this regard, literature has offered models for capturing structural changes in the volatility process. Schwert [17], for instance, considers a model in which returns' volatility is either high or low and allows parameters to shift between these two regimes in accordance with a Markov process with two states. To capture abrupt volatility fluctuations, Cai[3] constructed an ARCH model with parameters that switch regimes and later Gray [7] documented a Markov-switching GARCH model whose modification was proposed by Klaassen [12]. Bauwens et al.[2] created a regime-switching GARCH model that better describes the volatility behavior. This model provides for distinct parameters in each regime to account for the possibility that the data generation process undergoes a finite number of changes within the sample period. This model, according to Bauwens et al.[2], permits varied reversion speeds to different volatility levels at different intervals throughout the sample period. In terms of the regime-switching model, Liu et al.[14] used it to analyze option pricing. They looked at using a fast Fourier transform (FFT) approach to value options, in which the asset price is driven by a regime switching geometric Brownian motion. Hardy [11] developed a regime-switching model of long-term stock returns which she used to price the European options. Later on, Mitsui and Satoyoshi [16] applied the Markov switching GARCH model to price the Nikkei 225 options, assuming risk-neutral investors. Godin and Trottier [6] developed a regime switching framework with extended Girsanov principle to price options. In terms of pricing options, there is limited evidence that regime-switching and regime-switching GARCH models outperform the classic Black-Scholes model. As a result, the goal of this study is to develop a regime-switching model for European option valuation using the risk-neutral market assumption. The regime switching model is modified to include GARCH effects and dynamics resulting to regime-switching GARCH model, hereafter referred to as RS-GARCH. These two models are utilized to fit financial market data and the results compared with those from the famous Black-Scholes model.

The remaining part of this paper is arranged as follows: section 2 describes the methods used, including the derivation and estimation of the RS and RS-GARCH models. Section 3 deals with empirical data analysis and model implementation, while section 4 brings the paper to a close.

2. Methodology

2.1. **Regime-switching model.** A asset with no risk and a risky one tradable for a set period of time, $[t, T]$ are considered. Assume that $\{\alpha(t), t \geq 0\}$ is a continuous Markov process under the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{M} = \{1, 2, ..., k\}$ as the state space where $\mathbb{P}$ is the physical probability measure. In a regime switching market, let the price of stock, S_t, at time t satisfy the equation

$$dS_t = \mu_{\alpha(t)} S_t dt + \sigma_{\alpha(t)} S_t dW_t, \quad \alpha(t) = \{1, 2, ..., k\}, \tag{2.1}$$

such that the initial price, S_t is greater than zero, and the standard Brownian motion, W_t and $\alpha(t)$ are independent. The solution of equation (2.1) is determined through Itô lemma as

$$\ln(\frac{S_T}{S_t}) = (\mu_{\alpha(t)} - \frac{1}{2}\sigma^2_{\alpha(t)})\tau + \sigma_{\alpha(t)} W_\tau, \text{ for } \alpha(t) = \{1, 2, ..., k\}, \tag{2.2}$$

where $\tau = [T - t]$. The stock price undergoes discrete shifts between regimes $\alpha(t)$ and it is described by a Markov chain of order 1 and whose transition probability

P_{ij} from state i at time $t+1$ to state j at time t denoted by $P_{ij} = P\{\alpha(t+1) = j|\alpha(t) = i\}$ for all $i, j = \{1, 2, ..., k\}$. P_{ij} satisfies the conditions $0 \leq P_{ij} \leq 1$ and $\sum_{j=1}^{k} P_{ij} = 1$. The matrix of transition P_{ij} of the Markov chain is given by

$$P_{ij} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1k} \\ p_{21} & p_{22} & \cdots & p_{2k} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ p_{k1} & . & \cdots & p_{kk} \end{pmatrix}.$$

An equivalent measure $\mathbb{Q}$ is constructed from the risk neutral measure, in which the discounted stock price is a martingale. Equation (2.1) can thus be written as

$$\ln(S_T/S_t) = (r - \frac{1}{2}\sigma^2_{\alpha(t)})\tau + \sigma_{\alpha(t)} W_\tau, \text{ for } \alpha(t) = \{1, 2, ..., k\}. \tag{2.3}$$

Let $\mathcal{R}$ denote the total time spent in regime $\alpha(t) = j$ for $j = 1, 2, ..., k$ in the interval $[t, T]$ in n trials, given that at time t, the state is k. Denote the probability $Pr(\mathcal{R} = \alpha_j)$ by p for $j = 1, 2, ..., k-1$. For simplicity, we restrict ourselves to two regimes, i.e, $k = 2$, hence the transition matrix discussed earlier reduces to

$$P_{ij} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}.$$

As the number of transitions n grows larger, the fraction of time spent in each regime by the Markov chain can be calculated using a time-average(invariant) distribution of the Markov chain, that is, $\beta P = \beta$, where P is the transition matrix and β is the average fraction of time spent in state $\alpha(t) = j$ over n steps as n approaches infinity. Let $\beta = [\beta_1 \; \beta_2 \,]$, then

$$[\beta P] = [\beta_1 \; \beta_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = [\beta_1 \; \beta_2]. \tag{2.4}$$

This results into two equations as follows; $\beta_1 P_{11} + \beta_2 P_{21} = \beta 1 \text{and} \beta_1 P_{12} + \beta_2 P_{22} = \beta_2$ both of which simplify to $\beta_1 P_{12} = \beta_2 P_{21}($ since $P_{11} + P_{12} = 1$ and $P_{21} + P_{22} = 1$). Again, since β is a valid probability distribution, $\beta_1 + \beta_2 = 1$ and solving we get

$$\beta_1 = \frac{P_{21}}{P_{12} + P_{21}} \text{ and } \beta_2 = \frac{P_{12}}{P_{12} + P_{21}}. \tag{2.5}$$

Since, $\tau = T - t$ is the trading period, the entire amount of time the process spends in regimes 1 and 2 can now be computed as $\mathcal{R} = \beta_1 \tau$ and $\tau - \mathcal{R} = \tau - \beta_1 \tau$, respectively. In view of the research by Duan et al.[11] and Hardy[11] the distribution of log returns, $X_t = \log S_T - \log S_t$, conditional on the total time spent in regime $\alpha(t) = j$, for $j = 1, 2$, k can be developed such that there exist a normal density function whose mean and variance are μ^* , σ^{*2} , respectively. That is,

$$X_t|\mathcal{R} \; \sim \; N(\mu^*, \sigma^{*2}), \tag{2.6}$$

where $\mu^* = \frac{\mathcal{R}}{\tau}\mu_1 + \big(\frac{\tau-\mathcal{R}}{\tau}\big)\mu_2$ and $\sigma*^2 = \frac{\mathcal{R}}{\tau}\sigma_1^2 + \big(\frac{\tau-\mathcal{R}}{\tau}\big)\sigma_2^2$. Since p is the probability function for $\mathcal{R}$,

$$\begin{aligned} F_{X_t} = Pr\big[X_t \leq x\big] &= \sum_{j=1}^{k-1} Pr\big[X_t|\mathcal{R} = \alpha_j\big]p \\ &= \sum_{j=1}^{k-1} \phi_j\big(\frac{x-\mu*}{\sigma*}\big)p, \end{aligned} \tag{2.7}$$

where $\phi()$ is the standard normal probability distribution function. This implies that the probability density function for X_t is

$$f_{X_t} = \sum_{j=1}^{k-1} \phi_j\big(\frac{x-\mu*}{\sigma*}\big)p, \tag{2.8}$$

where $\phi()$ is the standard normal density function.

Now, define $C(K,T)$, the European call option(under regime switching world) with strike price K that matures after time T , and is valued at S_t at an initial time t. Since in a regime switching market, the parameter σ^2 switches regimes, we can define a parameter $\sigma*^2$ conditional on knowing $\mathcal{R}$, the total time spent in regime $\alpha(t) = j$ for $j = 1,2$. This implies that, the asset price $S_t|\mathcal{R}$ has a log-normal distribution with parameters that depend on $\mathcal{R}$, that is, the parameters are $\mu*$ and $\sigma*^2$ as defined earlier. Now, to derive a regime switching pricing model, the Black-Scholes formula is considered, and the parameter σ^2 is replaced with $\sigma*^2$ to give the desired model as follows;

$$C(K,T) = \mathbb{E}^{\mathbb{Q}}\big[max(X_T - K)|\mathcal{R}\big] = S_t\phi(d_1) - e^{-r\tau}K\phi(d_2) \quad \text{where} \tag{2.9}$$

$$d_1 = \frac{\ln(\frac{S_t}{K}) + r\tau + \frac{1}{2}\big[\mathcal{R}\sigma_1^2 + (\tau-\mathcal{R})\sigma_2^2\big]}{\sqrt{\mathcal{R}\sigma_1^2 + (\tau-\mathcal{R})\sigma_2^2}} \quad \text{and}$$

$$d_2 = d_1 - \sqrt{\mathcal{R}\sigma_1^2 + (\tau-\mathcal{R})\sigma_2^2}.$$

2.1.1. *Regime switching model parameter estimation.* According to Hamilton[8], the regime of a given process at time t is denoted by a random variable $\alpha(t)$ such that $\alpha(t) = 1,2,...,k$ where k is the maximum possible number of regimes. When the process X_t is in regime $\alpha(t) = j$ for $j = 1,2,...,k$, then it is presumed to have originated from a normal distribution with mean, μ_j and variance, σ_j^2 . Therefore, the density function of X_t conditional on $\alpha(t)$ taking on the value j is given by

$$f(X_t|\alpha(t) = j;\theta) = \frac{1}{\sigma_j\sqrt{2\pi}}\exp\Big\{-\frac{(X_t-\mu_j)^2}{2\sigma_j^2}\Big\}, \tag{2.10}$$

where $\theta = \{\mu_j, \sigma_j\}$ for $j = \{1,2,...,k\}$. The probability distribution presumed to have generated the unobserved regime $\alpha(t)$ for which the unconditional probability that $\alpha(t) = j$ is denoted by π_j , that is,

$$P\big\{\alpha(t) = j|\mathcal{F}_t;\theta\big\} = \pi_j \text{ for } j = \{1,2,...,k\}. \tag{2.11}$$

This implies that the vector θ now becomes $\theta = \{\mu_j, \sigma_j, \pi_j\}'$. The probability of the joint event that $\alpha(t) = j$ and X_t falls within some time interval [t,T]. This is determined by integrating

$$p(X_t, \alpha(t) = j;\theta) = f\big(X_t|\alpha(t) = j;\theta\big)P\big(\alpha(t) = j|\mathcal{F}_t;\theta\big) \tag{2.12}$$

over all values of X_t between t and T. The expression (2.12) is the joint density function of $\alpha(t)$ and X_t and utilizing equations (2.10) and (2.11), results to

$$p(X_t, \alpha(t) = j; \theta) = \frac{\pi_j}{\sigma_j\sqrt{2\pi}} \exp\Big\{ - \frac{\big(X_t - \mu_j\big)^2}{2\sigma_j^2}\Big\} \text{ for } j = \{1, 2, ..., k\}. \tag{2.13}$$

Note that summing equation (2.13) over all values for j gives

$$\begin{aligned} f(X_t; \theta) &= \sum_{j=1}^{k} p(X_t, \alpha(t) = j; \theta) \\ &= \frac{\pi_1}{\sqrt{2\pi}\sigma_1} \exp\Big\{ - \frac{\big(X_t - \mu_1\big)^2}{2\sigma_1^2}\Big\} + \ ... \ + \frac{\pi_j}{\sqrt{2\pi}\sigma_j} \exp\Big\{ - \frac{\big(X_t - \mu_j\big)^2}{2\sigma_j^2}\Big\}. \end{aligned} \tag{2.14}$$

Because the regime $\alpha(t)$ is unobserved, equation (2.14) best reflects the actually observed data X_t . The log likelihood for the observed data can be derived from Equation (2.14) if the regime variable $\alpha(t)$ is i.i.d across multiple dates, t, as

$$L(\theta) = \sum_{t=1}^{T} \log f(X_t; \theta). \tag{2.15}$$

The maximum likelihood of θ is determined by maximizing Equation (2.15) while keeping the constrains $\pi_1 + \pi_2 +, ..., +\pi_k = 1$ and $\pi_j \geq 0$ for $j = 1, 2, ..., k$. It is shown in Hamilton [8] that the parameters μ_j, σ_j^2 and π_j can be estimated as

$$\tilde{\mu_j} = \frac{\sum_{t=1}^{T} X_t P\{\alpha(t) = j | X_t; \tilde{\theta}\}}{\sum_{t=1}^{T} P\{\alpha(t) = j | X_t; \tilde{\theta}\}} \text{ for } j = 1, 2, ..., k. \tag{2.16}$$

$$\tilde{\sigma_j^2} = \frac{\sum_{t=1}^{T} (X_t - \mu_j)^2 P\{\alpha(t) = j | X_t; \tilde{\theta}\}}{\sum_{t=1}^{T} P\{\alpha(t) = j | X_t; \tilde{\theta}\}} \text{ for } j = 1, 2, ..., k. \tag{2.17}$$

$$\tilde{\pi_j} = T^{-1} \sum_{t=1}^{T} P\big\{\alpha(t) = j | X_t; \tilde{\theta}\big\} \text{ for } j = 1, 2, ..., k. \tag{2.18}$$

Furthermore, if we restrict the transition probability only by the conditions $P_{ij} > 0$ and $(P_{i1} + P_{i2} +, ..., +P_{ik}) = 1$ for all i and j, then Hamilton [9] documented the MLEs of the transition probability as

$$\tilde{P_{ij}} = \frac{\sum_{t=2}^{T} P\{\alpha(t) = j, \alpha(t-1) = j | X_t; \tilde{\theta}\}}{\sum_{t=2}^{T} P\{\alpha(t-1) = i | X_t; \tilde{\theta}\}} \text{ for } j = 1, 2, ..., k. \tag{2.19}$$

This implies that the predicted P_{ij} is simply the number of times state i appears to have been followed by state j divided by the number of times the process was in state i.

2.2. **Regime switching-GARCH model.** Let S_t be asset price at time t in a discrete-time economy. The one period asset returns under the physical measure $\mathbb{P}$ is defined as

$$\begin{aligned} X_t &= \ln S_t - \ln S_{t-1} \\ &= \mu_t + r_t, \ \ r_t = \sigma_t \varepsilon_t, \end{aligned} \tag{2.20}$$

where $\varepsilon_t \sim N(0,1)$. The general GARCH model is defined as

$$r_t = \sigma_t \varepsilon_t,\ \sigma_t^2 = \omega + \sum_{i=t}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{2.21}$$

where q is the GARCH degree; p is the ARCH process degree, ε_t is a set of zero-mean and unit-variance random variables that are distributed independently and identically. It is required that $\omega > 0,\ \alpha_i \geq 0,\ \beta_j \geq 0$ and $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$. It is argued by Bauwens [2] that estimation of this model using daily or higher frequency data implies that the volatility persistence is very high and the model may not be covariance-stationary. This persistence could be due to changes in the GARCH parameters over time, see Mikosch and Stărică, Cătălin [15], etc. To capture this regime swings, a Regime-Switching GARCH model is considered since it allows the parameters to shift regime. Define an unobserved state variable at time t as $S_t \in \{1, 2, .., k\}$ which selects the model parameters with probability $P_{ij} = P[\alpha(t) = j|\mathcal{F}_{t-1}]$ where $\mathcal{F}_{t-1}$ is the available information at time t. The RS-GARCH model can thus be defined as

$$\begin{aligned} X_t &= \mu_{\alpha(t)} + r_t,\ \text{where}\ r_t = \sigma_{t,\alpha(t)}\varepsilon_t\ \text{where}\ \varepsilon \sim N(0,1), \\ \sigma_{t,\alpha(t)} &= \omega_{\alpha(t)} + \sum_{i=t}^{p} \alpha_{i,\alpha(t)} r_{t-i}^2 + \sum_{j=1}^{q} \beta_k \sigma_{t-j}^2\ \text{ for }\ \alpha(t) = \{1, 2, ..., k\}. \end{aligned} \tag{2.22}$$

Let $\varphi(S_T)$ be the payoff of a European call option maturing at a future time T and whose exercise price is K, where $\varphi(S_T)$ is a random variable under probability measure space $(\Omega, \mathcal{F}, \mathbb{P})$ describing the market. Then, under the RS-GARCH(p,q) specification, the today's price of the call option C_t under measure $\mathbb{Q}$ is given by

$$\begin{aligned} C_t &= e^{-(T-t)r}\mathbb{E}^{\mathbb{Q}}\Big[\max(S_T - K, 0)|\mathcal{F}_t\Big] \\ &= e^{-(T-t)r}\mathbb{E}^{\mathbb{Q}}\Big[(S_T - K)^+|\mathcal{F}_t\Big]. \end{aligned} \tag{2.23}$$

For simplicity we restrict the model to RS-GARCH(1,1).

2.2.1. *RS-GARCH model parameter estimation.* The RS-GARCH model estimation is determined via the maximum likelihood estimation technique as reported by Ardia et al.[1]. The model is conditioned on normal distribution. Let $\theta = (\alpha_{0,k}, \alpha_{1,k}, \beta_k)'$ be the vector representing parameters of the model, then the likelihood function is given by

$$L(\theta|\mathcal{F}_T) = \prod_{t=1}^{T} f(r_t|\theta, \mathcal{F}_{t-1}). \tag{2.24}$$

Here, $f(r_t|\theta, \mathcal{F}_{t-1})$ is the probability density function of r_t given the previous observations, $\mathcal{F}_{t-1}$ and the model parameters θ. The conditional density of r_t, for RS-GARCH model is thus given by

$$f(r_t|\theta, \mathcal{F}_{t-1}) = \sum_{i=1}^{K}\sum_{j=1}^{K} P_{ij} z_{i,t-1} f_D(r_t|\alpha(t) = j, \theta, \mathcal{F}_{t-1}) \tag{2.25}$$

such that $z_{i,t-1} = P[\alpha(t-1) = j|\theta, \mathcal{F}_{t-1}]$ is the state i filtered probability at time $t-1$ computed via Hamilton filter, see Hamilton and Susmel[10]. In equation (2.25), the density of r_t in state $\alpha(t) = k$ given θ and $\mathcal{F}_{t-1}$ is expressed as

$f_D(r_t|\alpha(t) = k, \theta, \mathcal{F}_{t-1})$. The maximum likelihood estimate $\tilde{\theta}$ is thus determined by maximizing equation (2.24).

2.3. Root Mean Square Error (RMSE).

The Root Mean Square Error (RMSE) is determined using both the model's projected and observed option prices to compare the models in terms of option price prediction. In other words, the RMSE of a prediction model in relation to observed option prices is calculated using the square root of the mean squared error,

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n} (P_i - O_i)^2}{n}}, \tag{2.26}$$

where P_i is the predicted option prices, O_i the observed option prices and n is the sample size. Note that, lower values of the RMSE are an indication of a model with a better prediction.

3. Results and discussion

3.1. Empirical data.

The data utilized here for analysis is the daily closing price as reported in the Russell 2000 (RUT), Facebook (FB) and google(GooG) indices for the period January 2, 2013 to January 21, 2022. The indices returns, X_t , are computed as in equation (2.20). The options data utilized from the three markets is in two sets; that is, call options prices expiring in 25 and 258 days.

3.2. Descriptive statistics.

The plots of the stock returns and stock prices of Russell 2000, Facebook and Google series are presented in Figure 1. The three return indices shows the common properties of financial data, however the Facebook and Google indices returns have pronounced volatility clustering compared to the Russell 2000 index returns as displayed by long spikes. Table 1 presents the descriptive statistics for the three indices returns. The Russell 2000, Facebook and Google indices have a daily mean return of 0.0361%, 0.1045% and 0.0867%, respectively.

Table 1. **Basic statistics for indices returns**

Index	Obs	Mean	Var	Std dev.	Skew	Ex.Kurt	JB
RUT	2280	0.000361	0.000184	0.013571	-1.238128	13.110355	25291.3***
FB	2280	0.001045	0.000450	0.021220	0.403136	14.683610	29830.3***
GooG	2280	0.000867	0.000255	0.015979	0.419047	6.844511	9294.8***

The daily standard deviations are 1.3571%, 2.122% and 1.5979% respectively for the Russell 2000, Facebook and Google indices. Moreover, Russell 2000 reports a negative skewness of -1.238128 while Facebook and Google have a positive skewness of 0.403136 and 0.419047 respectively. The excess kurtosis is positive and higher than three for all the return series. These values imply that the returns are not distributed normally, that is, it has fat tails which is also confirmed by the Jarque-Bera(JB) statistic. The results are in tandem with properties of other financial returns.

3.3. **Empirical findings and discussion.** The parameter estimates of the RS model are estimated according to equation (2.9) and reported in Table 2. The parameters are different across the volatility regimes. For instance, the mean in low volatility regime are 0.0008, 0.0015 and 0.0013 for the Russell 2000, Facebook and Google indices returns respectively, which is higher than the corresponding mean in the higher volatility regime. In addition, in high volatility regimes, the mean are negative implying low returns and considerable risk in this regime. Moreover, the probability of switching regimes is low, that is, once the volatility process is in one regime it lingers before transiting to the next regime. The probability of transiting from low to high volatility regime is estimated at 0.0773, 0.2277 and 0.1931 for Russell 2000, Facebook and Google indices respectively.

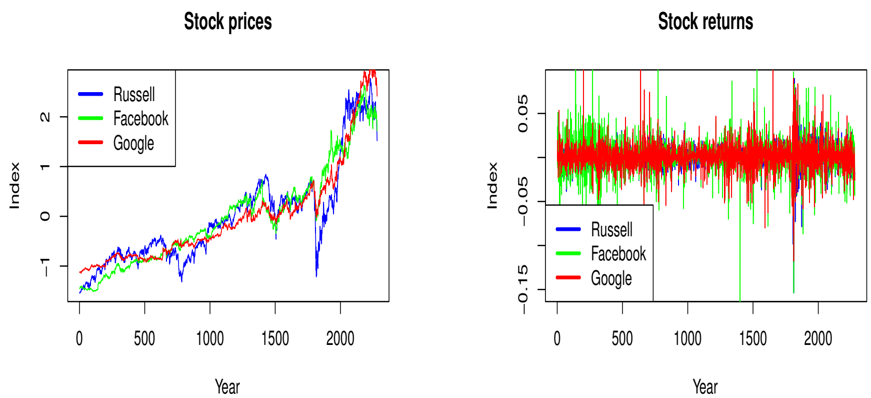

FIGURE 1. Stock prices and stock returns

This implies that once the process is in low volatility regime, it stays there for approximately 12, 4 and 5 days for Russell 2000, Facebook and Google indices respectively. However, under a high volatility regime, the process lasts longer than in a low volatility regime approximated at 79, 19 and 24 days for Russell 2000, Facebook and Google indices respectively. The RS-GARCH parameters are estimated

TABLE 2. **Regime-switching model parameter estimates**

Index	$\tilde{\mu_1}$	$\tilde{\mu_2}$	$\tilde{\sigma_1}$	$\tilde{\sigma_2}$	$\tilde{P_{12}}$	$\tilde{P_{21}}$
RUT	0.0008	−0.0002	0.0094	0.0273	0.0773	0.0126
FB	0.0015	−0.0009	0.0141	0.0394	0.2277	0.0515
GooG	0.0013	−0.0010	0.0105	0.0306	0.1931	0.0408

by utilizing equation (2.22) and reported in Table 3. Almost all the parameter estimates are significant at 5% significance level. These parameter estimates show that the volatility process is varied across regimes. Each volatility regime has different unconditional variances, which confirms the existence of different volatility regimes. The conditional mean estimates in high volatility regime, ω_i for $i = 1, 2$ are greater than the corresponding estimates in the low volatility regime across the three markets. Moreover, the volatility dynamics are implied by the ARCH term α_{1i} and GARCH term β_i for $i = 1, 2$. A large value of β_i indicates that the shock effects to future volatility take long to die off, that is, volatility is highly persistent whereas large values of α_{1i} display volatility reaction to the recent changes in price. A comparison of the two regimes shows that the low volatility regime has low values of the ARCH term but have high GARCH term. This means that the GARCH process is more reactive and less persistent in the low volatility regime than in high volatility regime. The persistence of volatility in each regime is calculated as $\alpha_{1i}+\beta_i$ for $i = 1, 2$ and it is required that $\alpha_{1i}+\beta_i < 1$ for covariance stationarity of the process. The calculated values for Russell 2000, Facebook and Google indices are; $\alpha_{11} + \beta_1 = 0.9633$ versus $\alpha_{12} + \beta_2 = 0.9764$, $\alpha_{11} + \beta_1 = 0.9855$ versus $\alpha_{12} + \beta_2 = 0.2641$ and $\alpha_{11} + \beta_1 = 1.0986$ versus $\alpha_{12} + \beta_2 = 0.1789$, respectively. It can be inferred that the low volatility regime for the Facebook and Google has high volatility persistence than the high volatility regime despite the process being explosive in the low volatility regime in the Google index. The volatility persistence is slightly higher in the high volatility regime than in low volatility regime for Russell 2000 index. The estimated option prices for some given strike prices from

Table 3. **RS-GARCH model parameter estimates**

Index	ω_1	ω_2	α_{11}	α_{12}	β_1	β_2	P_{12}	P_{21}
RUT	7.4×10^{-7}	3.6e-5	0.0683	0.3086	0.8950	0.6678	0.3527	0.8414
FB	1.0×10^{-7}	0.0045	0.0370	0.2639	0.9485	0.0002	0.0319	0.6405
GooG	2.2×10^{-6}	0.0010	0.1401	0.1785	0.9585	0.0004	0.0398	0.3072

the Russell 2000, Facebook and Google indices are computed based on equations (2.9) and (2.22) and presented in Table 4. The strike prices considered are for 25 and 258 days for the Russell 2000, Facebook and Google markets and the initial stock prices are 1987.92, 303.17 and 2601.84 respectively. It is assumed that the markets are without risk and an interest rate of 6% per annum is utilized. The computed call option prices are significantly different from the results reported by the Black-Scholes model and also the three models reports values that are slightly different from the actual market price values, however the price values from the Black-Scholes model seems to report prices that are close to the expected market prices as compared to the RS and RS-GARCH models. A comparison of the models is done by employing the Root Mean Square Error(RMSE) test and the results are presented in Table 5. The results portray RS-GARCH model as a better model than the Black-Scholes and RS model in estimating the 258 days option prices for Facebook and Google indices. On the other hand, for the 25 days option prices estimation the Black-Scholes model gives better estimates followed by RS-GARCH and RS models in that order.

Moreover, Figure 2 and 3 presents the plot of strike prices versus the estimated call options and the observed market option prices. Clearly, RS-GARCH model is revealed to give better estimates compared to the other models for the Facebook and Google stock market indices.

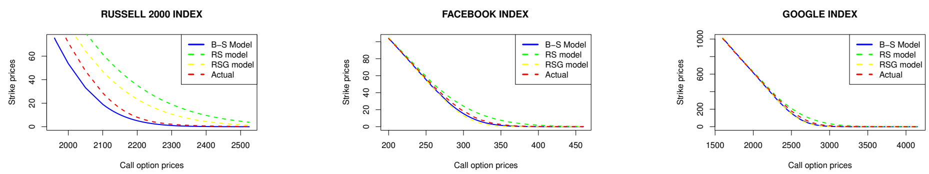

FIGURE 2. 25 days option prices

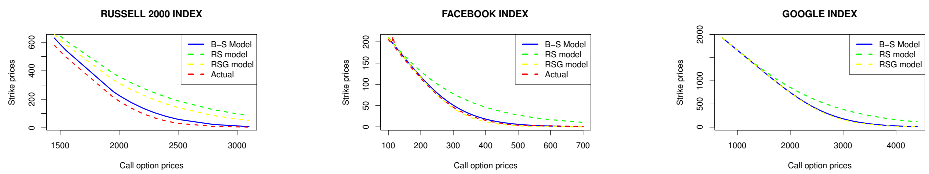

FIGURE 3. 258 days option prices

TABLE 4. **The Call Option prices**

		RUSSELL					FACEBOOK					GOOGLE		
		B-S	RS	RSG			B-S	RS	RSG			B-S	RS	RSG
Strike	Mkt	Call	Call	Call	Strike	Mkt	Call	Call	Call	Strike	Mkt	Call	Call	Call
25 days call option prices														
1960	94.2	75.36	121.47	106.22	200	103.83	104.36	104.50	104.36	1600	1005.35	1011.34	1011.35	1011.34
1965	91.2	72.40	118.82	103.50	260	47.55	45.57	49.95	30.79	1825	781.55	787.67	788.17	787.67
2000	71.4	53.70	101.37	85.74	280	31.55	28.50	35.65	21.39	2300	336.9	319.83	348.68	318.38
2050	47.35	33.12	79.71	64.23	285	27.85	24.78	32.52	31.34	2340	303.55	282.90	317.49	280.83
258 days call option prices														
1450	579.5	630.44	680.66	656.89	100	205.53	209.13	209.90	209.13	720	1892.15	1924.74	1924.98	1924.74
1500	537.5	586.32	645.00	618.56	105	200.85	204.43	205.42	204.42	740	1872.55	1905.93	1906.23	1905.93
1550	496	543.18	610.60	581.54	110	195.98	199.74	200.97	199.72	760	2045.55	1887.12	1887.49	1887.12
1950	214.5	254.11	382.18	337.00	115	186.40	195.04	196.57	195.02	780	2200.00	1868.31	1868.77	1868.31

TABLE 5. **Root Mean Square Error (RMSE)**

	25 days options			258 days options		
Index	BS	RS	RSG	BS	RS	RSG
RUT	5.159	21.48	11.65	29.78	160.0	115.4
FB	1.685	4.175	2.939	4.081	24.55	3.433
GooG	14.07	29.96	17.59	286.0	305.6	285.3

4. CONCLUSION

This paper focuses on developing a model with regime switch as well as extending it to incorporate GARCH in the regimes with the key purpose of pricing the European options. These models are utilized in pricing European options derived from the Russell 2000, Facebook and Google market indices. Two sets of data are utilized in fitting the model, that is, 25 and 258 days call option prices. The model comparison is carried out by computing the Root Mean Square Error (RMSE) for each model and the model with the least RMSE is the best model for pricing the European call options.

The results indicate that the financial time series for the three markets exhibit the common features of financial data such as volatility clustering, heavy tails among others. The parameter estimates of the models indicate that the market indices have distinct regimes. In this case two regimes are used, high and low volatility. It is clear that the low volatility regime has high volatility persistence for Facebook and Google data whereas the volatility process is explosive in low volatility regime for Google market index. The results show that RS-GARCH is the best model compared with Black-Scholes and RS models when applied to long-dated options contract. However, when short-dated options contract are used, Black-Scholes model out performs the RS and RS-GARCH models. Lastly, there is need for more empirical analysis to be carried out by other researchers so as to support our findings.

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