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ESTIMATION OF RELIABILITY FOR EXPONENTIAL CASE IN THE PRESENCE OF ONE OUTLIER

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Abstract. In the presence paper, we deal with the problem of estimating Reliability where Y has exponential distribution with parameter λ and X has exponential distribution with presence of one outlier with parameters θ and β, such that X and Y are independent. The moment, maximum likelihood and mixture estimators of Reliability are derived and has been shown that the moment estimator of Reliability is asymptotically unbiased estimator. At the end, we conclude that mixture estimators are better than the maximum likelihood and moment estimators.

1. Introduction

Consider spread from a point source, for example, which might a small plot of plants. During favourable weather conditions, the plants release their pollen and it disperses according to exponential distribution with distance from the source. However, in less favourable condition, light, rain or mist, not only are the plants less likely to release pollen, but that which is released still falls with an exponential distribution with a different scale parameter.

Consider the above example of a spread from a point source but now in the context of spread of disease amongst plants of viral spores such as barley yellow mosaic dwarf virus (BYMDV).

According to Dixit, Moore and Barnett (1996), we assume that a set of random variables $(X_1, X_2, ..., X_n)$ represent the distance of an infected sampled plant from a plot of plants inoculated with a virus. Some of the observations are derived from the airborne dispersal of the spores and are distributed according to the exponential distribution. The other observations out of n random variables (say k) are present because aphids which are known to be carriers of BYMDV have passed the virus into the plants when the aphids feed on the sap. These k aphids are considered to be exponentially distributed with changed scale.

In reliability contexts, inferences about R=P[Y¡X], where X and Y independent distributions, are a subject of interest. For example, in mechanical reliability of a system, if X is the strength of a component which is subject to stress Y, then R is a measure of system performance. The system fails, if at any time the applied

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stress is greater than its strength. Stress- strength reliability has been discussed in Kapur and Lamberson (1977). Some other aspects of inference about R are given in Al-Hussaini et al. (1997), Sathe and Dixit (2001) have been estimate of $R = P(Y < X)$ in the negative binomial distribution, and recently, Baklizi and Dayyeh have done shrinkage estimation of $R = P(Y < X)$ in exponential case.

Thus, we assume that the random variables $(X_1, X_2, ..., X_n)$ are such that one of them are distributed with p.d.f $f_1(x, \theta)$,

$$f_1(x,\theta) = \frac{\beta}{\theta} e^{-\frac{\beta x}{\theta}}, \qquad x > 0,\ \theta > 0,\ 0 < \beta \leq 1 \tag{1.1}$$

and the remaining $(n-1)$ random variables are distributed with p.d.f $f_2(x, \theta)$,

$$f_2(x,\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \qquad x > 0,\ \theta > 0,\ 0 < \beta \leq 1. \tag{1.2}$$

In section 2, 3 and 4, we deal with the methods of moment, maximum likelihood and mixture of methods of moment and maximum likelihood in estimating R. Analysis of a real life data set has been presented in section 5. We compare them together by Monte Carlo simulations and finally we draw conclusion in section 6.

2. Method of Moment

Let $Y_1, Y_2, ..., Y_n$ be a random sample for Y with pdf,

$$g(y,\lambda) = \frac{1}{\lambda} e^{-y/\lambda} \qquad , \qquad y > 0,\ \lambda > 0 \tag{2.1}$$

and $X_1, X_2, ..., X_n$ be random sample for X with pdf,

$$f(x,\theta,\beta) = \frac{1}{n}\frac{\beta}{\theta} e^{-\frac{\beta x}{\theta}} + \frac{n-1}{n}\frac{1}{\theta} e^{-\frac{x}{\theta}} \qquad , \qquad x > 0,\ \theta > 0,\ 0 < \beta \leq 1 \tag{2.2}$$

with presence of k (Know) outliers (see Dixit (1989) and Dixit and Nasiri (2001)). The parameter R we want to estimate is

$$\begin{aligned}
R &= P(Y < X) \\
&= \int_0^\infty \int_0^x g(y,\lambda) f(x,\theta,\beta)\, dy dx \\
&= \int_0^\infty \left[\int_0^x \frac{1}{\lambda} e^{-y/\lambda} dy\right] \left[\frac{1}{n}\frac{\beta}{\theta} e^{-\frac{\beta x}{\theta}} + \frac{n-1}{n}\frac{1}{\theta} e^{-\frac{x}{\theta}}\right] dx \\
&= \int_0^\infty (1 - e^{-x/\lambda}) \frac{1}{n}\frac{\beta}{\theta} e^{-\frac{\beta x}{\theta}} dx + \int_0^\infty (1 - e^{-x/\lambda}) \frac{n-1}{n}\frac{1}{\theta} e^{-\frac{x}{\theta}} dx \\
&= \frac{1}{n}\frac{\theta}{\lambda\beta + \theta} + \frac{n-1}{n}\frac{\theta}{\lambda + \theta}.
\end{aligned} \tag{2.3}$$

In case of no outlier presence, R was proposed by Baklizi and Dayyeh (2003). Since $P(Y < X) + P(Y > X) = 1$, we consider $P(Y > X)$,

$$R = P(Y > X) = \frac{b\lambda\beta}{\lambda\beta + \theta} + \frac{\bar{b}\lambda}{\lambda + \theta} \tag{2.4}$$

where $b = \frac{1}{n}$, $\bar{b} = \frac{n-1}{n}$ and $b + \bar{b} = 1$.

The moment estimator of R can be shown to be

$$\hat{R} = \frac{b\hat{\lambda}\hat{\beta}}{\hat{\lambda}\hat{\beta} + \hat{\theta}} + \frac{\bar{b}\hat{\lambda}}{\hat{\lambda} + \hat{\theta}} \tag{2.5}$$

where $\hat{\lambda} = \frac{1}{m}\sum_{i=1}^{m} y_i$, $\hat{\theta}$ and $\hat{\beta}$ can be obtained as;

From (4) we get

$$E(X) = \frac{b\theta}{\beta} + \bar{b}\theta, \tag{2.6}$$

and

$$E(X^2) = 2b\left(\frac{\theta}{\beta}\right)^2 + 2\bar{b}\theta^2. \tag{2.7}$$

Consider, $m_i^{'} = \frac{1}{n}\sum_{j=1}^{n} X_j^i$ and let $D = \frac{m_2^{'}}{m_1^{'}}$ and $H = \frac{m_2^{'}}{m_1^{'2}}$,

$$D = \frac{\frac{2b}{\beta^2} + 2\bar{b}}{\frac{b}{\beta} + \bar{b}} \cdot \theta, \tag{2.8}$$

and

$$H = \frac{\frac{2b}{\beta^2} + 2\bar{b}}{\left(\frac{b}{\beta} + \bar{b}\right)^2}. \tag{2.9}$$

From (11), we have

$$(\bar{b}^2 H - 2\bar{b})\beta^2 + 2b\bar{b}H\beta + (b^2H - 2b) = 0, \tag{2.10}$$

$$\xi_1\beta^2 - \xi_2\beta + \xi_3 = 0 \tag{2.11}$$

where

$$\begin{aligned} \xi_1 &= \bar{b}^2 H - 2\bar{b} \\ \xi_2 &= -2b\bar{b}H \\ \xi_3 &= b^2H - 2b \end{aligned}$$

if $\Delta = {\xi_2}^2 - 4\xi_1\xi_3$ is non-negative then the roots are real. Therefore

$$\hat{\beta} = \frac{\xi_2 + \sqrt{{\xi_2}^2 - 4\xi_1\xi_3}}{2\xi_1} \tag{2.12}$$

and, from (10)

$$\hat{\theta} = \frac{b\hat{\beta} + \bar{b}\hat{\beta}^2}{2b + 2\bar{b}\hat{\beta}^2} \cdot D \tag{2.13}$$

Thus, we can obtain the moment estimator of β and θ by using Equations (14) and (15), respectively.

Here, we shall show that $\hat{\theta}$ and $\hat{\beta}$ are asymptotically unbiased estimators. Let $W_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $W_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$, then $D = \frac{W_2}{W_1}$. Here, we can write $\hat{\theta}$ as a function of W_1, W_2,

$$\hat{\theta} = h(w_1, w_2). \tag{2.14}$$

Let

$$E(W_1) = \mu = \frac{b\theta}{\beta} + \bar{b}\theta,$$
$$E(W_2) = \nu = 2b\left(\frac{\theta}{\beta}\right)^2 + 2\bar{b}\theta^2.$$

Expand the function $h(w_1, w_2)$ around (μ, ν) by Taylor series,

$$\begin{aligned}\hat{\theta} &= h(w_1, w_2) \\ &= h(\mu,\nu) + (w_1 - \mu)\frac{\partial h}{\partial w_1}|_{w_1=\mu, w_2=\nu} + (w_2 - \nu)\frac{\partial h}{\partial w_2}|_{w_1=\mu, w_2=\nu} + \cdots \end{aligned} \quad (2.15)$$

then

$$\begin{aligned} E(\hat{\theta}) &= h(\mu,\nu) = \frac{b\beta + \bar{b}\beta^2}{2b + 2\bar{b}\beta^2} \cdot \frac{\nu}{\mu} \\ &= \frac{b\beta + \bar{b}\beta^2}{2b + 2\bar{b}\beta^2} \cdot \frac{2b\left(\frac{\theta}{\beta}\right)^2 + 2\bar{b}\theta^2}{b\left(\frac{\theta}{\beta}\right) + \bar{b}\theta} \\ &= \frac{b\beta + \bar{b}\beta^2}{2b + 2\bar{b}\beta^2} \cdot \frac{2b + 2\bar{b}\beta^2}{b\beta + \bar{b}\beta^2} \cdot \theta \\ &= \theta, \end{aligned}$$

and similarly,

$$E(\hat{\beta}) = \frac{b\theta}{b\left(\frac{\theta}{\beta}\right) + \bar{b}\theta - \bar{b}\theta} = \beta$$

Note that if we consider $W_3 = \frac{1}{m}\sum_{i=1}^{m} Y_i$ and $\hat{R} = h(w_1, w_2, w_3)$, It is easy to show that $E(\hat{R}) = R$.

3. Method of Maximum Likelihood

The maximum likelihood estimator of R can be shown to be,

$$\hat{R} = \frac{b\hat{\lambda}\hat{\beta}}{\hat{\lambda}\hat{\beta} + \hat{\theta}} + \frac{\bar{b}\hat{\lambda}}{\hat{\lambda} + \hat{\theta}} \quad (3.1)$$

where $\hat{\lambda} = \frac{1}{m}\sum_{i=1}^{m} y_i$ and to obtain the maximum likelihood estimator of θ and β, we consider the joint distribution of $\underline{X}$ with presence of k outliers,

$$L(\underline{x}, \theta, \beta) = \frac{\beta}{n\theta^n} e^{-\frac{n\bar{x}}{\theta}} \sum_{A=1}^{n} e^{\frac{(1-\beta)}{\theta} x_A} \quad (3.2)$$

For the more details see Dixit and Nasiri (2001).

If $L(\theta, \beta) = \ln(L(\underline{x}, \theta, \beta))$, then from (19),

$$L(\theta, \beta) = \ln\beta - n\ln\theta - \ln n - \frac{n\bar{x}}{\theta} + \ln\sum_{A=1}^{n} e^{\frac{(1-\beta)}{\theta} x_A}. \quad (3.3)$$

To solve for our MLEs of θ and β we take the derivative of the log likelihood ($L(\theta,\beta)$) with respect to each parameter, set the partial derivatives equal to zero, and solve for $\hat{\theta}$ and $\hat{\beta}$:

$$\frac{\partial L(\theta,\beta)}{\partial\theta} = \frac{-n}{\theta} + \frac{n\bar{x}}{\theta^2} - (1-\beta)\frac{\sum_{A=1}^{n} x_A e^{\frac{(1-\beta)}{\theta}x_A}}{\theta^2\sum_{A=1}^{n} e^{\frac{(1-\beta)}{\theta}x_A}} \overset{set}{=} 0 \tag{3.4}$$

$$\frac{\partial L(\theta,\beta)}{\partial\beta} = \frac{1}{\beta} - \frac{\sum_{A=1}^{n} x_A e^{\frac{(1-\beta)}{\theta}x_A}}{\theta\sum_{A=1}^{n} e^{\frac{(1-\beta)}{\theta}x_A}} \overset{set}{=} 0 \tag{3.5}$$

There is no closed-form solution to this system of equations, so we will solve for $\hat{\theta}$ and $\hat{\beta}$ iteratively, using the Newton-Raphson method, a tangent method for root finding. In our case we will estimate $\hat{\alpha} = (\hat{\theta},\hat{\beta})$ iteratively:

$$\hat{\alpha}_{i+1} = \hat{\alpha}_i - \mathbf{G}^{-1}\mathbf{g} \tag{3.6}$$

where $\mathbf{g}$ is the vector of normal equations for which we want

$$\mathbf{g} = [\ g_1 \quad g_2\ ]$$

with

$$g_1 = \frac{n\bar{x} - n\theta}{\theta^2} - \frac{(1-\beta)\sum_{A=1}^{n} x_A e^{\frac{(1-\beta)}{\theta}x_A}}{\theta^2\sum_{A=1}^{n} e^{\frac{(1-\beta)}{\theta}x_A}} \tag{3.7}$$

$$g_2 = \frac{1}{\beta} - \frac{\sum_{A=1}^{n} x_A e^{\frac{(1-\beta)}{\theta}x_A}}{\theta\sum_{A=1}^{n} e^{\frac{(1-\beta)}{\theta}x_A}} \tag{3.8}$$

and $\mathbf{G}$ is the matrix of second derivatives

$$\mathbf{G} = \begin{bmatrix} \frac{dg_1}{d\theta} & \frac{dg_1}{d\beta} \\ \frac{dg_2}{d\theta} & \frac{dg_2}{d\beta} \end{bmatrix} \tag{3.9}$$

where

$$\frac{dg_1}{d\theta} = \frac{2(n\theta - n\bar{x}) - n\theta}{\theta^3} + \frac{(1-\beta)^2}{\theta^4}\left(\frac{\Upsilon_3}{\Upsilon_1} + \frac{{\Upsilon_2}^2}{{\Upsilon_1}^2} - \frac{2\,\theta\Upsilon_2}{(1-\beta)\Upsilon_1}\right) \tag{3.10}$$

$$\frac{dg_1}{d\beta} = \frac{dg_2}{d\theta} = \frac{(1-\beta)}{\theta^3}\left(\frac{\Upsilon_3}{\Upsilon_1} - \frac{{\Upsilon_2}^2}{{\Upsilon_1}^2} + \frac{\theta\Upsilon_2}{(1-\beta)\Upsilon_1}\right) \tag{3.11}$$

$$\frac{dg_2}{d\beta} = \frac{-1}{\beta^2} + \frac{1}{\theta^2}\left(\frac{\Upsilon_3}{\Upsilon_1} - \frac{{\Upsilon_2}^2}{{\Upsilon_1}^2}\right) \tag{3.12}$$

with

$$\begin{aligned}
\Upsilon_1 &= \sum_{A=1}^{n} e^{\frac{(1-\beta)}{\theta} x_A} \\
\Upsilon_2 &= \sum_{A=1}^{n} x_A e^{\frac{(1-\beta)}{\theta} x_A} \\
\Upsilon_3 &= \sum_{A=1}^{n} (x_A)^2 e^{\frac{(1-\beta)}{\theta} x_A}
\end{aligned}$$

The Newton-Raphson algorithm converges, as our estimates of θ and β change by less than a tolerated amount with each successive iteration, to $\hat{\theta}$ and $\hat{\beta}$. In case of no outlier presence, $\hat{\theta}$ was proposed by Baklizi and Dayyeh (2003).

4. Mixture of Methods of Moment and Maximum Likelihood

Read (1981) proposed the methods, which avoid the difficulty of complicated equations. According to Read (1981), replacement of some, but not all, of the equations in the system of likelihood may make it more manageable. From (15), we have

$$\hat{\theta} = \frac{b\hat{\beta} + \bar{b}\hat{\beta}^2}{2b + 2\bar{b}\hat{\beta}^2} \cdot D \tag{4.1}$$

and, From (22)

$$\hat{\beta} = \frac{\hat{\theta} \sum_{A=1}^{n} e^{\frac{(1-\hat{\beta})}{\hat{\theta}} x_A}}{\sum_{A=1}^{n} x_A e^{\frac{(1-\hat{\beta})}{\hat{\theta}} x_A}}. \tag{4.2}$$

5. Numerical Study

In order to have some idea about Bias and Mean Square Error (MSE) of MLE, Moment and Mixture methods, we perform sampling experiments using a MATLAB. The simulation study was carried out for $\theta = 0.2$, $\beta = 0.3$ and $\lambda = 0.4$ with samples size ; (n,m)=(15,15),(20,20),(25,25), (15,20), (20,15), (15,25), (25,15), (20,25), (25,20). Here we present a complete analysis of a simulated data. The data has been generated using $m = n = 30$, $\theta = 0.2$, $\beta = 0.3$ and $\lambda = 0.4$, therefore $R = 0.6375$.

The Y values are

0.3074	0.2746	0.8443	0.0039	0.0510	0.8618	0.7102	0.2112	0.2880	1.3477
0.1517	0.2024	0.1734	0.4999	0.4457	0.6604	1.3119	0.0078	0.1725	0.2355
0.3584	0.0400	0.1583	0.1848	0.0407	0.4378	1.0296	0.3667	0.1951	1.3288

and the corresponding X values are

0.8348	0.1759	0.6817	0.0095	0.0846	0.0182	0.2837	0.1013	0.1309	0.0733
0.0258	0.2118	0.2236	1.2749	0.3482	0.0467	0.2433	0.1590	0.5222	0.0976
0.1059	0.5936	0.0850	0.0107	0.1346	0.1140	0.7411	0.0536	1.8191	0.1989

Now, we obtain the MLE, Mom and Mix estimates of λ, θ, β and R as; $\hat{\lambda}_{MLE} = 0.4300$, $\hat{\theta}_{MLE} = 0.2740$, $\hat{\beta}_{MLE} = 0.3545$, and therefore $\hat{R}_{MLE} = 0.5855$. Also, $\hat{\lambda}_{Mom} = 0.4300$, $\hat{\theta}_{Mom} = 2536$, $\hat{\beta}_{Mom} = 0.2978$, and therefore $\hat{R}_{Mom} = 0.5997$. Then $\hat{\lambda}_{Mix} = 0.4300$, $\hat{\theta}_{Mix} = 0.2536$, $\hat{\beta}_{Mix} = 0.3545$ and therefore $\hat{R}_{Mix} = 0.6037$. We see that the mixture estimator of R is better than the maximum likelihood and moment estimators.

6. Conclusions

In this paper, we have addressed the problem of estimating $P(Y < X)$ for the Exponential distribution with presence of one outlier. The moment, maximum likelihood and mixture estimators of R are derived and has been shown that the moment estimator of R is asymptotically unbiased estimator. All the results are base on 1000 replications and are given in Table 1. In this case as expected when $m = n$ and m , n increase then the average biases and the MSEs decrease. For fixed m as n increase the MSEs decrease and also for fixed n as m increases the MSEs decrease.

From Tables 1, we conjecture that the moment estimator of R is asymptotically unbiased. On the other hand, the moment and mixture estimators are underestimation, but the maximum likelihood estimator is overestimation. The MSEs of any three estimators are tending to zero, and when $m = n$ and m , n increase then the MSEs decrease, and for fixed m as n increase the MSEs decrease and also for fixed n as m increases the MSEs decrease. Table 1 show that the mixture estimator have the smallest estimated MSEs as compared with the moment and maximum likelihood estimators. We strongly feel mixture estimators are better and easy to calculate than the maximum likelihood and moment estimations. From the previous observations, we suggest to use mixture method for estimating $R = P(Y < X)$ in exponential case with presence of one outlier because it is easy to calculate than the rest.

Table: Tables 1 give Bias and MSE when outlier come from exponential distribution.

Table 1

Bias and MSE for : $\beta = 0.3$, $\theta = 0.2$, $\lambda = 0.4$

		Bias			MSE		
(n,m)	R	MLE	Mom	Mix	MLE	Mom	Mix
(15,15)	0.6472	0.0054	-0.0137	-0.0180	0.0074	0.0075	0.0080
(20,20)	0.6521	0.0046	-0.0081	-0.0101	0.0052	0.0053	0.0051
(25,25)	0.6550	0.0093	-0.0069	-0.0038	0.0039	0.0040	0.0036
(15,20)	0.6472	0.0070	-0.0094	-0.0096	0.0057	0.0065	0.0053
(20,15)	0.6521	-0.0017	-0.0075	-0.0084	0.0063	0.0064	0.0062
(15,25)	0.6472	0.0194	-0.0065	-0.0097	0.0059	0.0059	0.0058
(25,15)	0.6550	-0.0035	-0.0068	-0.0089	0.0054	0.0056	0.0052
(20,25)	0.6521	0.0026	-0.0070	-0.0174	0.0043	0.0049	0.0039
(25,20)	0.6550	0.0103	-0.0078	-0.0199	0.0040	0.0046	0.0039

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References

[1] Al-Hussaini. E. Mousa, M. Sultan, K., Parametric and nonparametric estimation of $P(Y < X)$ for finite mixtures of Lognormal components. *Communications in statistics - Theroy and Methods, (1997), 26(5): 1269-1289.*

[2] Baklizi, A. and Dayyh, W. A., Shrinkage estimation of $P(Y < X)$ in the exponential case. *Communications in statistics - Theory and Methods, (2003), 32(1) : 31-42.*

[3] Dixit, U. J. , Estimation of parameters of the gamma distribution in the presence of outliers. *Communications in statistics - Theory and Methods, (1989), 18, 3071-3085*

[4] Dixit, U. J., Moore, K. L. and Barnett, V. , On the estimation of the power of the scale parameter of the exponential distribution in the presence of outliers generated from uniform distribution, *Metron, (1996), 54, 201-211.*

[5] Dixit, U. J. and Nasiri, P. F., Estimation of parameters of the exponential distribution in the presence of outliers generated from uniform distribution, *Metron, (2001), 49(3-4) : 187-198.*

[6] Kapur, K. C. and Lamberson, L. R. , Reliability in engineering design. *John Wiley and sons, New York, (1977).*

[7] Read, R. R. , Representation of certain covariance matrices with application to asymptotic efficiency. *Journal of the American Statistical Associate, (1981), 76, 148 - 154*

[8] Sathe, Y. S. and Dixit, U. J. , Estimation of $(X \leq Y)$ in negative binomial distribution. *Journal of Statistical Planning and Inference, (2001), 93, 83-92.*

[9] Schafft, H. A., Station, T. c., Mandel, J. and Shott, J. D. , Reproducibility of electromigration measurements, *IEEE Transactions on Electron Devices, (1987), ED-34, 673-681.*

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